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PRINCIPAL EFFECTS OF AXIAL LOAD ON MOMENT-DISTRIBUTION ANALYSIS OF RIGID STRUCTURES

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PRINCIPAL EFFECTS OF AXIAL LOAD ON MOMENT-DISTRIBUTION ANALYSIS OF RIGID STRUCTURES*

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SUMMARY

This thesis presents the method of moment distribution modified to include the effect of axial load upon the bending moments. This modification makes it possible to analyze accurately complex structures, such as rigid fuselage trusses, that heretofore had to be analyzed by approximate formulas and empirical rules. The method is simple enough to be practicable even for very complex structures, and it gives a means of analysis for continuous beams that is simpler than the extended three-moment equation now in common use.

When the effect of axial load is included, it is found that the basic principles of moment distribution remain unchanged, the only difference being that the factors used, instead of being constants for a given member, become functions of the axial load. Formulas have been developed for these factors, and curves plotted so that their application requires no more work than moment distribution without axial load. Simple problems have been included to illustrate the use of the curves.

INTRODUCTION

The importance of saving weight in airplane structures makes it necessary accurately to consider the secondary moments caused by the combination of axial load and lateral deflection. Formulas considering the secondary moments in the case of continuous beams are quite familiar

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to the aeronautical engineer. They were originally derived by Müller-Breslau and have been extended by Professor J. S. Newell and presented in chapter XI of reference 1. However, no similar practical method has been hitherto available for the analysis of complex rigid frames when the members are subjected to axial load.

Before the method of moment distribution was developed, rigid frame analysis presented a very difficult problem. In building design it was the usual practice to use approximate formulas, necessitating very conservative assumptions for the sake of safety. Least work, slope deflection, and other similar methods based on the principle of consistent deformations, but neglecting the secondary moments due to axial load, were used when it was necessary to get a more accurate solution. These methods all involve the solution of simultaneous equations, however, and when the degree of redundancy is high, the number of equations involved necessitates very tedious computations. As these methods are too complex for practical use, it would hardly be worth while to complicate them further by including the effects of axial load. However, the development of moment distribution in the last few years has given a means of rigid building frame analysis that is simple enough to be practicable for complex as well as simple structures. If this could be combined with Newell's equations, without an excessive sacrifice of simplicity, the result would be very valuable to the aeronautical engineer. This thesis is the record of what is believed to be a satisfactory and practical solution of the problem of combining these two methods of analysis.

As the Newell formulas have been used by aeronautical engineers for several years, it will be assumed that the reader is familiar with their use; they will not be discussed here. The method of moment distribution is relatively new, however, and there has been very little standardization of nomenclature and sign convention. For this reason a brief review of the basic principles will be given.

Moment distribution was first presented by Professor Hardy Cross in an article entitled "Analysis of Continuous Frames by Distributing Fixed-End Moments", published in the May 1930 issue of the Proceedings of the A.S.C.E. The article has been reprinted, together with all the discussion that followed as reference 2. Professor Cross has also included a thorough discussion of the method in reference 3.

Considerable interest has been attracted by the simplicity of the method with the result that several articles have been written for the purpose of presenting briefly its more important elements. A paper by Harry A. Williams (reference 4) presented as a thesis at Stanford University (later modified as reference 5), gives a very clear presentation of the fundamental principles and includes numerous examples that aid in understanding the application of the method. A brief discussion is presented by E. F. Bruhn (reference 6) in Aviation Engineering of March 1933. None of these papers, however, considers the effect upon the bending moments when axial load is present in the members of the frame.

The fundamental principle of the method of moment distribution is the assumption that at first a fictitious condition exists in the structure; this condition is then modified, step by step, until the condition that actually exists is reached. The initial fictitious condition is that all the joints of the structure are rigidly fixed against rotation, or "locked." In this condition the external loads create easily computed bending moments at the ends of each span that is transversely loaded. The algebraic sum of all these "fixed-end moments" at any joint constitutes an unbalanced moment that tends to rotate that joint. Under the hypothetical assumption that all the joints are "locked," however, no rotation actually takes place. One of the joints is now assumed "unlocked" and allowed to rotate under the influence of its unbalanced moment until a resisting moment is built up that brings the joint into equilibrium. The effect of this balancing moment upon the stresses of the member is computed, and the joint is "locked" again.

When a joint is unlocked, there are two distinct effects upon the structure. First a moment equal and opposite to the unbalanced moment at the joint is added. Physically this moment is created by the resistance to rotation of each member coming into the joint. Thus each member contributes a part of this resisting moment and, as all the members rotate through the same angle, it has been shown that the contribution of each member is directly proportional to its "stiffness factor." For a member with constant moment of inertia and without axial load this stiffness factor is equal to EI/L.

The second effect of unlocking a joint is the addition of a moment at the far end of each member. Assuming positive moments as those acting on the end of a member in a clockwise direction, the convention that will be used throughout this paper, this moment is of the same sign and equal to the moment at the near end times a "carry-over factor." For members with constant moment of inertia and no axial load, the carry-over factor is 0.5.

The process of unlocking and locking the joints, one at a time, is continued until all the joints have been unlocked, balanced, locked again, and the carry-over moments recorded. As each joint is unlocked, the effect on the bending moments of the structure is computed. It will now be found that some of the joints that have been balanced and relocked have become unbalanced again, due to the carry-over moments from other joints. The process must therefore be repeated, these joints being balanced again and new carry-over moments recorded. This procedure is continued until the unbalanced moments and carry-over moments are small enough to be neglected. If all the joints are now unlocked simultaneously, the effect on the bending moments of the structure will be negligible. The moments at the ends of the members, therefore, are the same as those that would have existed if the structure had been allowed to deflect directly, instead of step by step. These moments may be found by totaling the fixed-end moments, the moments distributed to the member each time the joint was unlocked, and the moments carried over to that end from the other end of the member. It is not necessary to continue the process until the unbalanced moments completely disappear. The operations may be stopped and the moments totaled whenever the desired degree of accuracy, as indicated by the magnitude of the unbalanced moments, is reached.

When axial load, either tension or compression, is present in the members of a frame, the secondary moments due to the combination of axial load and deflection alter the fundamental method of moment distribution to no greater degree than the ordinary three-moment equation is modified in the extended equation. The distribution factors, carry-over factors, and fixed-end moments, instead of being constant for a given member, become functions of L/j.

The principal purpose of this thesis is to develop a method of rigid frame analysis that combines Newell's formulas with the Hardy Cross method, and present it in a form that may be easily used by the engineer. In so doing the following steps have been taken:

1. The formulas for carry-over factor, stiffness factor, and fixed-end moments in terms of L/j have been derived.

2. A method of considering joint translation has been developed.

3. Curves have been plotted to make the use of the formulas practical.

4. Simple numerical examples have been given to illustrate the use of the curves and show how the method may be used as a simplification of the extended three-moment equation.

5. An example of an airplane fuselage, the members of which are subjected to both transverse and axial loads, has been given to show how the method may be applied to complex structures that heretofore have been impossible to analyze accurately.

When applying the method to an actual problem, the first values that are used are the fixed-end moments, next the stiffness, or distribution factors, of the members, and finally the carry-over factors. It might seem more logical to develop the formulas for these quantities in this order; however, the derivations are simpler if they are treated in the opposite order. This procedure should offer no confusion to anyone familiar with the principles of moment distribution.

In the development of the formulas, the same general methods of procedure are followed as were used by Professor Cross in his original derivations except that the effect of axial load has been included.

The writer wishes to express his thanks to Professor A. S. Niles for suggesting the subject and for his helpful advice and valuable assistance in the development of the thesis.

CARRY-OVER FACTOR

Assume a beam as shown in figure 1, rigidly supported at B and pinned at A. A is free to rotate, but restrained from transverse motion. The axial load P is assumed as compression. With a given moment M_A applied at A, it is desired to find the magnitude of the resisting moment at B, M_B.

This problem is most readily solved by the use of the extended three-moment equation. Assume the beam consists of two spans: The left span AB is of length L, and the right span has zero length.* Then using the three-moment equation:

$$\frac{M_1 L_1 \alpha_1}{I_1} + 2M_2 \left[\frac{L_1 \beta_1}{I_1} + \frac{L_2 \beta_2}{I_2}\right] + \frac{M_3 L_2 \alpha_2}{I_2} = 0$$

$$M_1 = M_A$$

$$M_2 = M_B$$

$$M_3 = 0$$

$$\frac{M_A L \alpha}{I} + 2\ M_B \left(\frac{L\beta}{I} + 0\right) + 0 = 0$$

or

$$M_A \alpha + 2\ M_B \beta = 0$$

$$M_B = -\ \frac{\alpha}{2\beta}\ M_A \tag{1}$$

It should be noted that all moments have been assumed positive when causing compression in the upper fibers of the beam. This is the sign convention used in the three-moment equation. As the convention used for moment distribution assumes that positive moments are those acting on a beam in a clockwise direction, the sign of M_A is the same for both systems. However, when M_B is positive in one system, it is negative in the other. Hence using the moment distribution convention of signs, equation (1) becomes:

$$M_B = +\ \frac{\alpha}{2\beta}\ M_A \tag{2}$$

*It is demonstrated on page 62 of reference 1 that if one end of a member without axial load is rigidly fixed against rotation, the moment at that end may be found by using the three-moment equation, with zero length of one of the spans. When axial load is present in the member, the same line of reasoning may be followed, showing that the extended three-moment equation may be used to find the moment at the fixed end.

Expressed in words, this equation states that when a moment M_A is applied at A, a moment equal to $(\alpha/2\beta)M_A$ is built up at B. Thus the carry-over factor of a member rotated at one end and rigidly supported at the other is $\alpha/2\beta$. This expression is plotted against L/j in graph I. It is evident that when $\frac{L}{j} = 0$, the condition when no axial load exists, the carry-over factor is 0.5, agreeing with the usual factor of the Hardy Cross method.

When the axial load is tension, the carry-over factor becomes $\alpha_h/2\beta_h$, which should be apparent from the similarity of the three-moment equations for compression and tension. The derivation is similar to that for compression, the hyperbolic functions being substituted for the circular. The derivation is given in the Appendix, and the equation is plotted in graph I along with the curve for compression.

In the Appendix is given a second proof of equation (2). Instead of using the extended three-moment equation, the more basic principle of moment areas is employed.

STIFFNESS FACTOR

In the last section it has been shown that $M_B = -(\alpha/2\beta)M_A$, using the sign convention of the precise equations as given in reference 1. Considering positive rotations as clockwise, the angle through which point A rotates is the negative of the slope at A as given by formula on page 201 of reference 1.

$$\therefore \theta_A = -i = -\frac{1}{P}\left[\frac{M_2-M_1}{L} - \frac{M_2-M_1 \cos\frac{L}{j}}{j \sin\frac{L}{j}} \cos\frac{x}{j} + \frac{M_1}{j} \sin\frac{x}{j}\right]$$

where

$$M_1 = M_A$$

$$M_2 = M_B = -\frac{\alpha}{2\beta} M_A$$

$$x = 0$$

$$\theta_A = -\frac{1}{P}\left[\frac{\left(-\frac{\alpha}{2\beta}\right) M_A - M_A}{L} + \frac{M_A \frac{\alpha}{2\beta} + M_A \cos\frac{L}{j}}{j \sin\frac{L}{j}}\right]$$

$$= -\frac{M_A}{Pj}\left[\frac{\alpha \csc\frac{L}{j}}{2\beta} + \cot\frac{L}{j} - \frac{\frac{\alpha j}{2\beta} + j}{L}\right]$$

From the equations on page 212 of reference 1, the values of $\cot\frac{L}{j}$ and $\csc\frac{L}{j}$ have been found in terms of α and β and substituted in this expression, giving:

$$\theta_A = -\frac{M_A}{Pj^2}\left[\frac{L\alpha^2}{12\beta} + \frac{j^2\alpha}{2\beta L} + \frac{j^2}{L} - \frac{L\beta}{3} - \frac{\alpha j^2}{2\beta L} - \frac{j^2}{L}\right]$$

$$= \frac{M_A L}{3EI}\left[\beta - \frac{\alpha^2}{4\beta}\right]$$

or

$$M_A = \frac{3EI\ \theta_A}{L\left(\beta - \frac{\alpha^2}{4\beta}\right)} = 4E\ \theta_A\ \frac{I}{L}\left(\frac{3\beta}{4\beta^2 - \alpha^2}\right)$$

When a joint of a rigid structure is rotated, all the members coming into the joint rotate through the same angle. Hence θ_A is the same for all the members. Assuming homogeneity of material, E is also the same for all members. Hence the moment required to rotate a joint is divided among all the members in proportion to the K values of each member, where

$$K = \frac{I}{L}\left(\frac{3\beta}{4\beta^2 - \alpha^2}\right) \tag{3}$$

The expression $\frac{3\beta}{4\beta^2 - \alpha^2}$ will be called the "stiffness-factor coefficient" and is plotted against L/j in graph II. To find the stiffness factor for a given member, determine the coefficient for the appropriate value of L/j from the curve and multiply by I/L of the member. When $\frac{L}{j} = 0$, the condition of no axial load, the coefficient is 1.0, giving a stiffness factor of I/L.

In case a joint of a structure is pinned, and it is thus known that the final moment at the ends of all members joining there must equal zero, it is a waste of time to alternately lock and unlock the joint. A more direct method is to treat this type of member as a special case, unlocking it after the first cycle and leaving it unlocked thereafter. (Unless the member has no fixed-end moments, it must be considered locked during the first cycle, or the fixed-end moment formulas would have to be modified, a complication that is not justified.)

Once a pinned joint is unlocked and left unlocked, it need not be considered further in the computations, as it is balanced, and no moments can be carried over to it, for it is incapable of developing a resisting moment when the far end is rotated. This means that when one end of a member is pinned, the carry-over factor to that pinned joint is zero.

It requires a smaller moment to rotate one end of a member through a given angle if the far end of that member is pinned than if it is fixed. Hence the stiffness factor of a member with one end pinned is less than it would be if that end were fixed. Consequently, the formula for stiffness factor (equation (3)) does not apply when one end of the beam is pinned, and a different formula must be developed. When finding the value of θ_A in the derivation of equation (3), it was assumed that the far end of the member B was rigidly supported, and hence that $M_B = -\left(\frac{\alpha}{2\beta}\right) M_A$. In case the far end is pinned, $M_B = 0$ and the derivation is accordingly modified:

$$\theta_A = -\frac{1}{P}\left[-\frac{M_A}{L} + \frac{M_A \cos\frac{L}{j}}{j \sin\frac{L}{j}}\right] = -\frac{M_A}{Pj}\left[\cot\frac{L}{j} - \frac{j}{L}\right]$$

$$= -\frac{M_A}{P}\left[\frac{j}{L} - \frac{L\beta}{3j} - \frac{j}{L}\right] = \frac{M_A\, L\beta}{3EI}$$

$$M_A = 4E\ \theta_A\ \frac{I}{L}\left(\frac{3}{4\beta}\right)$$

The term $4E\ \theta_A$ is the same for all members coming into the joint, whether fixed or pinned at the far end, and so the stiffness factor of a member whose far end is pinned becomes:

$$K_p = \frac{I}{L}\left(\frac{3}{4\beta}\right) \tag{4}$$

The coefficient $\frac{3}{4\beta}$ has been plotted in graph II along with the coefficient of members with the far end fixed. When $\frac{L}{j} = 0$, $\beta = 1$ and $\frac{3}{4\beta} = \frac{3}{4}$, giving a stiffness factor of $\frac{3}{4} \times \frac{I}{L}$, which agrees with the factor used for this type of member when axial load is neglected.

When the axial load is tension, the derivation of the formulas is similar, the only difference being the substitution of the hyperbolic functions for the circular. The stiffness factors for the two cases are therefore:

$$K = \frac{I}{L}\left(\frac{3\beta_h}{4\beta_h^2 - \alpha_h^2}\right) \quad \text{for far-end rigid,} \tag{5}$$

and

$$K_p = \frac{I}{L}\left(\frac{3}{4\beta_h}\right) \quad \text{for far-end pinned.} \tag{6}$$

These expressions have also been plotted in graph II.

In the Appendix the formulas for stiffness factor are derived by the method of moment-areas. As a further check the special case of a two-span beam rigidly supported at both ends and free to rotate at the center support, with an external moment applied at the center support, is solved by the extended three-moment equation. Solving for the division of moment between the two spans gives the same formulas as derived above.

FIXED-END MOMENTS

A fixed-end moment is the moment that exists at the ends of a loaded member when those ends are rigidly fixed against rotation. The most important loading conditions are: uniformly distributed load over all or part of the span, uniformly varying load, and a concentrated load at any point on the span. The formulas for these conditions are developed in this paper and, as the principle of superposition applies,* the fixed-end moments due to any combination of these loads may be found by adding the moments due to the separate loads.

The extended three-moment equation is used in deriving the equations for fixed-end moments. A beam of three spans is considered, the two end spans being of zero length, and the equations solved for the moments over the two inner supports. These moments are the desired fixed-end moments.

In case the axial load is tension, the derivation of the formulas is parallel to that for compression. The only difference is the use of the hyperbolic rather than the circular functions. The resulting formulas are the same as those for compression except that $\sin L/j$, $\cos L/j$, $\tan L/j$, etc. are changed to $\sinh L/j$, $\cosh L/j$, $\tanh L/j$, etc., and α, β, and γ are changed to $-\alpha_h$, $-\beta_h$, and $-\gamma_h$, respectively.**

In all cases the curve for fixed-end moment when the axial load is tension has been plotted either on the same sheet as the curve for axial compression or on the next following sheet.

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*It is shown on pages 199 and 209 of reference 1 that, as long as the axial load remains constant, the total moment at any point on a beam is given by adding the separate moments of all the individual transverse loads on the beam. It is only when the axial load is varied that the principle of superposition does not apply.

**The formula for α_h is: $\alpha_h = -\dfrac{6\left(\frac{L}{j}\,\operatorname{csch}\frac{L}{j} - 1\right)}{\left(\frac{L}{j}\right)^2}$. It may be seen that this formula is the same as that for α as given on page 212 of reference 1 except that $\csc L/j$ has been changed to $\operatorname{csch} L/j$, and the formula preceded by a minus sign. Consequently, in substituting for α in any formula when changing to the hyperbolic form, it is necessary to use $-\alpha_h$. The same is true of the formulas for β_h and γ_h.

FIXED-END MOMENTS FOR A UNIFORM LOAD OVER ENTIRE SPAN

In figure 2 is shown a beam rigidly supported at A and B and subjected to a uniformly distributed load of w pound per inch. The extended three-moment equation for spans 1-2 and 2-3 is:

$$\frac{M_1 L_1 \alpha_1}{I_1} + 2M_2 \left(\frac{L_1 \beta_1}{I_1} + \frac{L_2 \beta_2}{I_2}\right) + \frac{M_3 L_2 \alpha_2}{I_2} = \frac{wL_2^3 \gamma_2}{4I_2}$$

where

$$M_1 = M_4 = 0$$
$$M_2 = M_A$$
$$M_3 = -M_B$$
$$L_1 = L_3 = 0$$
$$L_2 = L$$
$$I_2 = I$$

this becomes

$$2M_A\ L\beta - M_B\ L\alpha = \frac{wL^3 \gamma}{4}$$

Similarly, using spans 2-3 and 3-4:

$$M_A\ L\alpha - 2M_B\ L\beta = \frac{wL^3 \gamma}{4}$$

Solving these two simultaneous equations:

$$M_A = -\ M_B = \frac{wL^2}{(4/\gamma)\ (2\beta + \alpha)}$$

The expression $[(4/\gamma)\ (2\beta + \alpha)]$ has been plotted against L/j in graph III. To determine the fixed-end moments for a given beam with uniformly distributed load of w pound per inch, divide wL^2 by the coefficient found in graph III. When w is an up load, the left-hand moment M_A will be positive and the right-hand moment M_B will be negative.

It is apparent from the curve that when $\frac{L}{j} = 0$, the coefficient $[(4/\gamma)\ (2\beta + \alpha)] = 12$. Thus the fixed-end moments are equal to $wL^2/12$, which is the formula used when axial load is neglected.

EXAMPLE

Before proceeding to the derivation of the other fixed-end moment formulas, it seems advisable to give an example showing the application of the formulas already developed.

A symmetrical, three-support, continuous beam with cantilever overhangs will be used. A drawing of the beam with a uniformly distributed load of 10 pounds per inch is shown in figure 3. An axial load of compression is assumed to be of such magnitude that $\frac{L}{j} = 2.5$ for each of the spans BC and CD.

The first step is to compute the fixed-end moments. $M_{F_{BA}}$ is the moment created by the cantilever overhang. This moment is equal to $\frac{wL^2}{2} = \frac{10(30)^2}{2} = 4{,}500$. As the load is an up load, a counterclockwise moment at B is necessary to prevent rotation of AB. Hence the sign is negative, and $M_{F_{BA}} = -\ 4{,}500$ in.-lb. Similarly, $M_{F_{DE}} = 4{,}500$ in.-lb.

From the compression curve of graph III, it is seen that the fixed-end moment coefficient for $\frac{L}{j} = 2.5$ is equal to 10.69. Hence the fixed-end moments are:

$$M_{F_{BC}} = M_{F_{CD}} = \frac{wL^2}{C} = \frac{10(100)^2}{10.69}$$

$$M_{F_{CB}} = M_{F_{DC}} = -\ 9{,}355 \text{ in.-lb.}$$

Next the stiffness factors must be computed. As AB is a cantilever, M_{BA} can never have any value except -4,500. Hence any unbalanced moment at joint B must be resisted entirely by BC. That is, the stiffness factor of AB = 0. Similarly, the stiffness factor of DE is also zero. Owing to the symmetry of the beam, it is unnecessary to compute the stiffness factors of BC and CD, as they are obviously equal, and any unbalanced moment at C will be divided equally between CB and CD.

From graph I it is found that the carry-over factor of BC or CD is equal to 0.731. For convenience, this is entered in figure 4 at the center of each span.

The values of fixed-end moments are now written directly below the beam, as in figure 4. It is apparent that there is an unbalanced moment of 9,355 - 4,500 = 4,855 in.-lb. at joint B and one of -4,855 in.-lb. at joint D. Unlocking joint B first, it is necessary to add a balancing moment of -4,855 in.-lb. This moment is all distributed to BC, as the stiffness factor of AB is zero. A line is drawn under the -4,855 to indicate that the joint is in balance. The carry-over moment of joint C may be recorded now, or this step may be delayed until after joints C and D have been balanced, and then all of the carry-over moments recorded at once. The latter method is usually the simpler. Consequently, joint C is balanced next. As there is no unbalanced moment at this joint, the balancing moments will be zero. Joint D is next balanced, a moment of 4,855 in.-lb. being distributed to CD and 0 in.-lb. to DE.

All the joints are now in balance, except for the carry-over moments, which should now be recorded. The -4,855 in.-lb. distributed to BC is carried over as -4,855 X 0.731 = -3,549 in.-lb. to C. Similarly 3,549 in.-lb. are carried from D to C. The carry-over moments from C to B and from C to D are zero, as no moment was distributed at joint C.

As the carry-over moments to joint C were equal and opposite, and as no moments were added to joints B or D, all the joints are still in balance. All three joints may now be unlocked simultaneously without any effect upon the structure, as there are no unbalanced moments at any joint. The structure is therefore in equilibrium without the necessity of any hypothetically locked joints, and the moments at the ends of the members may be found by totaling:

$$M_{BA} = -4,500$$

$$M_{BC} = +9,355 - 4,855 = 4,500 \text{ in.-lb.}$$

$$M_{CB} = -9,355 - 3,549 = -12,904 \text{ in.-lb.}$$

$$M_{CD} = +9,355 + 3,549 = 12,904 \text{ in.-lb.}$$

$$M_{DC} = -9,355 + 4,855 = -4,500 \text{ in.-lb.}$$

$$M_{DE} = +4,500 \text{ in.-lb.}$$

The extended three-moment equation gives a value for $M_C = 12,903$ in.-lb.

FIXED-END MOMENTS FOR A UNIFORM VARYING LOAD

A uniformly varying load is shown in figure 5. The three-moment equations for this loading are:

$$2M_A\ L\beta - M_B\ L\alpha = wj^2L\ (\alpha - 1)$$

$$M_A\ L\alpha - 2M_B\ L\beta = 2wj^2L\ (\beta - 1)$$

Solving these for M_A and M_B:

$$M_A = \frac{wL^2}{\left(\frac{L}{j}\right)^2 \left[\frac{4\beta^2 - \alpha^2}{2(\alpha - \beta)}\right]}$$

$$M_B = -\frac{wL^2}{\left(\frac{L}{j}\right)^2 \left[\frac{4\beta^2 - \alpha^2}{4\beta^2 - \alpha^2 + \alpha - 4\beta}\right]}$$

The denominators of the right-hand side of these equations are plotted in graph IV. The corresponding coefficients for axial tension are given in graph V.

If the maximum load is at the left end of the beam, the formulas for M_A and M_B are reversed numerically, although in all cases the left-hand moment is positive and the right hand negative under positive loads.

FIXED-END MOMENTS FOR A CONCENTRATED LOAD AT MIDSPAN

The loading is shown in figure 6. The three-moment equations for this loading are:

$$2M_A\ L\beta - M_B\ L\alpha = 6Wj^2 \left[\frac{\sin\frac{L}{2j}}{\sin\frac{L}{j}} - \frac{1}{2}\right]$$

$$M_A\ L\alpha - 2M_B\ L\beta = 6Wj^2 \left[\frac{\sin\frac{L}{2j}}{\sin\frac{L}{j}} - \frac{1}{2}\right]$$

Solving for M_A and M_B:

$$M_A = -M_B = \frac{WL}{\left(\frac{L}{j}\right)^2 \left[\frac{2\beta + \alpha}{3\left(\sec\frac{L}{2j} - 1\right)}\right]}$$

The denominator of the right-hand side of this expression has been plotted in graph VI.

FIXED-END MOMENTS FOR A CONCENTRATED LOAD AT ANY POINT ON THE SPAN

The dimensions are shown in figure 7. The three-moment equations for this type of load are:

$$2M_A\ L\beta - M_B\ L\alpha = 6Wj^2 \left[\frac{\sin\frac{b}{j}}{\sin\frac{L}{j}} - \frac{b}{L}\right]$$

$$M_A\ L\alpha - 2M_B\ L\beta = 6Wj^2 \left[\frac{\sin\frac{a}{j}}{\sin\frac{L}{j}} - \frac{a}{L}\right]$$

Solving for M_A and M_B:

$$M_A = \frac{6WL\left[2\beta\left(\dfrac{\sin\frac{b}{L}\frac{L}{j}}{\sin\frac{L}{j}} - \dfrac{b}{L}\right) - \alpha\left(\dfrac{\sin\frac{a}{L}\frac{L}{j}}{\sin\frac{L}{j}} - \dfrac{a}{L}\right)\right]}{\left(\frac{L}{j}\right)^2(4\beta^2 - \alpha^2)}$$

$$M_B = \frac{6WL\left[\alpha\left(\dfrac{\sin\frac{b}{L}\frac{L}{j}}{\sin\frac{L}{j}} - \dfrac{b}{L}\right) - 2\beta\left(\dfrac{\sin\frac{a}{L}\frac{L}{j}}{\sin\frac{L}{j}} - \dfrac{a}{L}\right)\right]}{\left(\frac{L}{j}\right)^2(4\beta^2 - \alpha^2)}$$

When $\frac{a}{L} = 0.5$, these equations reduce to those given for the particular case of the load at midspan.

In order to use a larger scale amd hence improve the accuracy of the readings, the expressions for M_A and M_B have been plotted as the ratio of M_A and M_B to the fixed-end moments that would exist at A and B respectively if the axial load were zero.

When there is no axial load, the formulas for fixed-end moment for this type of loading are:

$$M_{A_o} = WL\left(\frac{a}{L}\right)\left(\frac{b}{L}\right)^2$$

$$M_{B_o} = -WL\left(\frac{a}{L}\right)^2\left(\frac{b}{L}\right)$$

The curves of graph VII give the ratio $C_A = M_A/M_{A_o}$. C_A is plotted against a/L, a being the distance from the left end of the beam to the load. Curves are drawn for several values of L/j. In order to find M_A, the value of C_A must be interpolated between the curves. Straight-line interpolation will give a maximum error of less than 2 percent, and in most instances the error will be less than 1 percent. If greater accuracy than this is necessary, the formulas may be used.

When C_A has been found, M_A may be found by the expression:

$$M_A = WL \left(\frac{a}{L}\right) \left(\frac{b}{L}\right)^2 C_A$$

To find the value of M_B, use the same curves but read the value of the coefficient for b/L rather than a/L as before. Call this coefficient C_B.

Then $$M_B = - WL \left(\frac{a}{L}\right)^2 \left(\frac{b}{L}\right) C_B$$

For convenience in finding the values of M_A and M_B after the coefficients C_A and C_B have been determined, a curve of $\left(\frac{a}{L}\right) \left(\frac{b}{L}\right)^2$ has been plotted against a/L in graph VIII. To find the value of $(a/L)^2 (b/L)$, use the same curve but read the value for b/L rather than a/L as before.

For example, suppose it is desired to find the fixed-end moments for a beam 100 inches long loaded with 1,000 pounds at a point 70 inches from the left end of the beam. The beam is shown in figure 8. L/j is assumed to be 3.8 in compression.

$$\frac{a}{L} = \frac{70}{100} = 0.7$$

$$\frac{b}{L} = \frac{30}{100} = 0.3$$

From graph VII at $\frac{a}{L} = 0.7$:

$$C_A = 1.625 - \frac{4.0 - 3.8}{4.0 - 3.5} (1.625 - 1.406) = 1.537$$

From graph VIII at $\frac{a}{L} = 0.7$, $\left(\frac{a}{L}\right) \left(\frac{b}{L}\right)^2 = 0.063$:

$$\therefore M_A = 1{,}000 \times 100 \times 0.063 \times 1.537 = 9{,}670 \text{ in.-lb.}$$

From graph VII at $\frac{b}{L} = 0.3$:

$$C_B = 1.389 - \frac{0.2}{0.5}(1.389 - 1.260) = 1.337$$

From graph VIII at $\frac{b}{L} = 0.3$, $\left(\frac{a}{L}\right)^2 \left(\frac{b}{L}\right) = 0.147$:

$$\therefore M_B = -1{,}000 \times 1{,}000 \times 0.147 \times 1.337 = -19{,}640 \text{ in.-lb.}$$

FIXED-END MOMENTS FOR A UNIFORM LOAD OVER PART OF SPAN

The loading is shown in figure 9. The three-moment equations are:

$$2M_A L\beta - M_B L\alpha = 3wj^2 \left[\frac{b^2 - L^2}{L} + 2j\left(\frac{\cos\frac{a}{j} - 1}{\tan\frac{L}{j}} + \sin\frac{a}{j}\right)\right]$$

$$M_A L\alpha - 2M_B L\beta = 3wj^2 \left[-\frac{a^2}{L} + \frac{2j\left(1 - \cos\frac{a}{j}\right)}{\sin\frac{L}{j}}\right]$$

Solving for M_A and M_B:

$$M_A = \frac{3wL^2}{\left(\frac{L}{j}\right)^2 (4\beta^2 - \alpha^2)} \left(2\beta\left\{\left(\frac{b}{L}\right)^2 - 1 + \frac{2\left[\left(\cos\frac{a}{L}\frac{L}{j}\right) - 1\right]}{\frac{L}{j}\tan\frac{L}{j}} + \frac{2\sin\frac{a}{L}\frac{L}{j}}{\frac{L}{j}}\right\}\right.$$

$$\left. - \alpha\left\{\frac{2\left[1 - \left(\cos\frac{a}{L}\frac{L}{j}\right)\right]}{\frac{L}{j}\sin\frac{L}{j}} - \left(\frac{a}{L}\right)^2\right\}\right)$$

$$M_B = \frac{3wL^2}{\left(\frac{L}{j}\right)^2 (4\beta^2-\alpha^2)} \left(\alpha\left\{\left(\frac{b}{L}\right)^2 - 1 + \frac{2\left[\left(\cos\frac{a}{L}\frac{L}{j}\right) - 1\right]}{\frac{L}{j}\tan\frac{L}{j}} + \frac{2\sin\frac{a}{L}\frac{L}{j}}{\frac{L}{j}}\right\}\right.$$

$$\left. - 2\beta\left\{\frac{2\left[1 - \left(\cos\frac{a}{L}\frac{L}{j}\right)\right]}{\frac{L}{j}\sin\frac{L}{j}} - \left(\frac{a}{L}\right)^2\right\}\right)$$

When there is no axial load, the formulas for fixed-end moments for this type of loading are:

$$M_{A_0} = \frac{wL^2}{12}\left(\frac{a}{L}\right)^2\left[6 - 8\frac{a}{L} + 3\left(\frac{a}{L}\right)^2\right] = wL^2 \; C_{A_0}$$

$$M_{B_0} = -\frac{wL^2}{12}\left(\frac{a}{L}\right)^2\left[4\frac{a}{L} - 3\left(\frac{a}{L}\right)^2\right] = -wL^2 \; C_{B_0}$$

The curves of graphs IX and X give the ratios $C_A = M_A/M_{A_0}$ and $C_B = M_B/M_{B_0}$ plotted against a/L for several values of L/j. The curves are plotted from $a/L = 0$ to $a/L = 0.5$. To find the fixed-end moments when a/L is greater than 0.5, find the fixed-end moments for a uniform load over the entire span and subtract the fixed-end moments that would be caused by a load over the part of the span that is not loaded.

For convenience in computation, curves of C_{A_0} and C_{B_0} have been plotted in graph VIII.

As an example, suppose it is desired to find the fixed-end moments of a beam such as that shown in figure 10. The span is 100 inches and is loaded uniformly with 10 pounds per inch for a distance of 70 inches. L/j is assumed at 3.6 in compression.

First find the fixed-end moments due to a positive load of 10 pounds per inch uniformly distributed over the entire span. This moment is found by the use of graph III.

$$M_A = - M_B = \frac{wL^2}{C} = \frac{10(100)^2}{9.11} = 10{,}980 \text{ in.-lb.}$$

Next find the fixed-end moments due to a <u>negative</u> load of 10 pounds per inch extending over the 30 inches from C to B. In this case the load is on the right side of the beam, whereas the curves of graphs IX and X apply to loads extending out from the left side. Hence, the curves must be reversed and C_A used in finding M_B and C_B in finding M_A.

From graph IX we find for $\frac{a}{L} = 0.03$:

$$C_A = 1.171 + \frac{3.6 - 3.5}{4.0 - 3.5} (1.253 - 1.171) = 1.187$$

From graph VIII at $\frac{a}{L} = 0.3$, $C_{A_o} = 0.0290$

$$M_B = - wL^2 \, C_{A_o} \, C_A = +10(100) \times 0.0290 \times 1.187 = 3{,}440 \text{ in.-lb.}$$

From graph X for $\frac{a}{L} = 0.3$, we find:

$$C_B = 1.396 + \frac{0.1}{0.5} (1.613 - 1.396) = 1.439$$

From graph 8 for $\frac{a}{L} = 0.3$, $C_{B_o} = 0.0070$

$$M_A = + wL^2 \, C_{B_o} \, C_B = -10(100) \times 0.0070 \times 1.439 = - 1{,}007 \text{ in.-lb.}$$

The net moments at A and B due to the load of 10 pounds per inch extending over 70 percent of the span are:

$$M_A = +10{,}980 - 1{,}007 = +9{,}973 \text{ in.-lb.}$$

$$M_B = -10{,}980 + 3{,}440 = -7{,}540 \text{ in.-lb.}$$

SECOND EXAMPLE OF CONTINUOUS BEAM ANALYSIS

Figure 11 shows the left half of a beam that rests on five supports. The beam and loads are symmetrical about support D. It is assumed that there is axial compression in BC and CD giving $\frac{L}{j} = 3$ for both these members. The value of I is constant for the entire beam.

Joint D may be assumed rigidly fixed against rotation and never unlocked, as any fixed-end moment or moment being carried to it is always exactly balanced by the symmetrical moment on the other half of the beam. Hence there is never any unbalanced moment at D, and no need to unlock it.

In this problem the work has been carried to much greater accuracy than necessary in order to show the agreement with the extended three-moment equation. The values of stiffness factor, carry-over factor, and fixed-end moments have been taken from the tables, where they may be obtained with more precision than can be read from the curves.

From graph I or table B, the carry-over factor for $\frac{L}{j} = 3$ is found to be 0.91893. From graph II or table B, the stiffness factor of BC, which has a pinned end at B, is found to be 0.10206 and that for CD is 0.65605. Hence any unbalanced moment at C will be distributed 0.10206/(0.10206 + 0.65605) = 13.462 percent to BC and 86.538 percent to CD.

The fixed-end moments are found as follows:

$$M_{F_{BA}} = +50 \times 100 = +5{,}000 \text{ in.-lb.}$$

$$M_{F_{BC}} = -500 \times 80 \times 0.144 \times 1.2135 = -6{,}989.8 \text{ in.-lb.}$$

$$M_{F_{CB}} = +500 \times 80 \times 0.096 \times 1.2590 = +4{,}834.6 \text{ in.-lb.}$$

$$M_{F_{CD}} = -\frac{10(80)^2}{24.560} = -2{,}605.9 \text{ in.-lb.}$$

$$M_{F_{DC}} = +\frac{10(80)^2}{17.072} = +3{,}748.8 \text{ in.-lb.}$$

The operations involved in unlocking and locking the joints are indicated in figure 12. First the fixed-end moments are recorded as shown. Next joint B is balanced by adding 1,989.8 in.-lb. to BC, as the stiffness factor of the cantilever is zero. The moment of 1,989.8×0.91893= 1,828.5 in.-lb. is immediately carried over to C. Joint C now has an unbalanced moment of 4,834.6+1,828.5-2,605.9 = 4,057.2 in.-lb. This moment is balanced by a moment of -4,057.2 in.-lb. distributed to CB and CD; -4,057.2× 0.13462 = -546.2 in.-lb. to CB and the remainder, -3,511.0 in.-lb. to CD. As BC is pinned at B, no moment is carried over to B. However, -3,511.0×0.91893 = -3,226.4 in.-lb. must be carried over to D. All the joints are now in balance, joint D being balanced by the equal and opposite moments from the other half of the beam, and the moments at any joint may be found by totaling.

Considering bending moments positive when the upper fibers are in compression, $M_B = -5{,}000$ in.-lb., $M_C = -6{,}116.9$ in.-lb., and $M_D = -522.4$ in.-lb. A solution by the extended three-moment equation gives values of $-5{,}000$, $-6{,}116.8$, and -522.5 in.-lb. for M_B, M_C, and M_D, respectively.

THIRD CONTINUOUS BEAM EXAMPLE

When an unsymmetrical beam of four or more supports, or a symmetrical beam of six or more supports, or a rigid frame of three or more members is analyzed, the moments do not become zero after the first cycle, as in the example of figure 11, but the process must be repeated until the unbalanced moments are small enough to neglect.

Figure 13 shows half of a symmetrical beam resting on seven supports and loaded at the end of the cantilever so that the moment at $A = 1{,}000$ in.-lb., the top fibers being in tension. The spans are equal in length and the moment of inertia is constant throughout. Axial compression is assumed of such value that $\frac{L}{j} = 3$ for AB, BC, and CD. This beam, without axial load, has been analyzed in references 4 and 5, and the results will be compared with those that include axial load to show the importance of secondary moments in continuous beams subjected to high compressive loads. A solution of this problem by the extended three-moment equation requires the solution of three simultaneous equations.

The carry-over factor is 0.9189 for all spans, found from graph I or table B. The stiffness factor of AB is found from the "far-end pinned" curve of graph II or table B and is equal to 0.10206. The stiffness factors of BC and CD are found from the "far-end restrained" curve of graph II or table B and are both equal to 0.65605. At B the distribution factors are 0.10206/(0.10206 + 0.65605) = 13.46 percent to BA and 86.54 percent to BC. At C the distribution factors are 50 percent to each member. These are designated by the symbol D and are recorded at the joints in a space provided as shown in figure 14. The carry-over factors are indicated by the symbol C and are written at the center of each member as shown. The only fixed-end moment is at the cantilever and is equal to 1,000 in.-lb.

Joint A is balanced first, -1,000 in.-lb. being added to AB, and -1,000 × 0.9189 = -918.9 in.-lb. carried over to B. This leaves an unbalanced moment at B which is balanced by 918.9 in.-lb. distributed 918.9 × 13.46 percent = 123.7 in.-lb. to BA and 795.2 in.-lb. to BC. As A is a pinned joint, no moment can be carried over to it, but 795.2 × 0.9189 = 730.6 in.-lb. must be carried over to C. This unbalanced moment at C is balanced by -365.3 in.-lb. to both CB and CD. Carry-over moments of -365.3 × 0.9189 = -335.7 in.-lb. are recorded at B and D.

The moment at joint D is balanced by the similar moment from the other half of the beam, but it must be noticed that joint B is no longer in balance, having had -335.7 in.-lb. carried over to it since it was balanced. This moment must therefore be balanced, the balancing moment distributed to BA and BC, the proper moments carried over, and the process continued until the desired accuracy is reached. Figure 14 shows the computations carried through nine cycles, and the totals are indicated. The total after nine more cycles is also recorded, and the values given by the extended three-moment equation are given as a check. The values when axial load is neglected, as given by Williams' results, are also recorded. It should be noted that all the moments except that at the cantilever are many times as large when axial load is considered as when it is neglected. The reason for this difference is that a high value of L/j was used. If a low value of L/j, say 1.0 or 1.5, had been used, the agreement between the two methods would have been much better.

EFFECT OF JOINT TRANSLATION

There are two types of joint translation that will be considered in this paper. In the first type the amount of translation is known, as on a continuous beam with a known or assumed deflection of one or more of the supports. In the second type the amount of translation is unknown but the total shear on a given section is known, as on a rectangular bent subjected to side loads.

When the amount of deflection is known, formulas for considering the effect of the deflection upon the moments of the structure may be derived by the extended three-moment equation. When the amount of deflection is unknown, a method that was developed by Professor Clyde T. Morris of Ohio State University (reference 7), for the case when axial load is neglected, is modified to include the effect of axial load.

JOINT TRANSLATION - AMOUNT OF TRANSLATION KNOWN

The translation of one or more joints of a rigid structure modifies the bending moments throughout. In both the basic and extended three-moment equation the effect of translation of one or more of the supports is determined by adding deflection terms to the load terms of the equation. These deflection terms are the same whether the basic or extended equation is used.

In moment-distribution analysis, deflection of the joints creates additional fixed-end moments. Figure 15 shows a beam rigidly supported at both ends. B is deflected an amount δ above A. It is desired to find the moments M_A and M_B that exist at the ends of the beam. The extended three-moment equation for spans 1-2 and 2-3 is:

$$\frac{M_1 L_1 \alpha_1}{I_1} + 2M_2 \left(\frac{L_1 \beta_1}{I_1} + \frac{L_2 \beta_2}{I_2}\right) + \frac{M_3 L_2 \alpha_2}{I_2} = \frac{6E(y_1 - y_2)}{L_1} + \frac{6E(y_3 - y_2)}{L_2}$$

Where

$$M_1 = M_4 = 0$$

$$M_2 = M_A$$

$$M_3 = -M_B$$

$$L_1 = L_3 = 0$$

$$y_1 - y_2 = 0$$

$$y_3 - y_2 = \delta$$

this becomes:

$$\frac{2M_A L\beta}{I} - \frac{M_B L\alpha}{I} = \frac{6E\delta}{L}$$

Similarly, using spans 2-3 and 3-4:

$$\frac{M_A L\alpha}{I} - \frac{2M_B L\beta}{I} = -\frac{6E\delta}{L}$$

Solving for M_A and M_B:

$$M_A = M_B = \frac{6E\delta I}{L^2}\,\frac{1}{(2\beta - \alpha)}$$

or

$$M_A = M_B = \frac{6K\ E\ R}{2\beta - \alpha}$$

where $K = \frac{I}{L}$ and $R = \frac{\delta}{L}$

If the equation is written

$$M_A = M_B = -\frac{6K\ E\ R}{2\beta - \alpha} \qquad (7)$$

then R is positive when the deflection is such that the member is rotated in a clockwise direction from its original location. The fixed-end moments due to joint deflection have the same sign at both ends of the span, both having the sign opposite to that of R. The quantity $2\beta - \alpha$ in equation (7) is a function of L/j and has been plotted in graph XI.

EXAMPLE OF CONTINUOUS BEAM WITH DEFLECTION OF SUPPORTS

Figure 11 shows the left half of a symmetrical continuous beam resting on five supports. This problem was previously analyzed assuming no deflection of the supports. This same beam will now be considered assuming that support C deflects 0.8 inch downward. I is constant at 0.2 in.^{4}, E = 29,000,000 lb./sq.in., and L/j is assumed equal to 3.0, as before.

$K = \frac{I}{L} = \frac{0.2}{80} = 0.0025$ for both BC and CD. As the deflection of joint C tends to rotate BC in a clockwise and CD in a counterclockwise direction. $R_{BC} = \frac{\delta}{L} = \frac{0.8}{80} =$ 0.01 and $R_{CD} = -0.01$. The fixed-end moments due to deflection of C are therefore:

$$M_{F_{BC}} = M_{F_{CB}} = -\frac{6KER}{2\beta-\alpha} = -\frac{6\times0.0025\times29000000\times0.01}{1.1915} = -3650.9$$

$$M_{F_{CD}} = M_{F_{DC}} = 3650.9 \text{ in.-lb.}$$

In figure 16 these moments have been added below the fixed-end moments caused by the loads. The remainder of the solution is similar to that when there is no deflection of the supports, and is recorded in figure 16. The bending moment at C is found to be 5,369.3 in.-lb. and that at D, 1,505.4 in.-lb., both with the lower fibers in compression. The extended three-moment equation also gives values of 5,369.3 and 1,505.4 in.-lb. for the moments at C, and D, respectively.

SECONDARY MOMENTS IN TRUSSES DUE TO JOINT TRANSLATION

When a rigid joint truss is subjected to a system of external loads, the individual members are stressed by axial tension or compression. As a result, each member is elongated or shortened by an amount equal to PL/AE. This change in length of the members causes the displacement of the joints of the structure, with the result that bending moments are developed at the ends of the members. The value of R for each member can be determined by the use of a Williot diagram or the cotangent formulas as explained in Art. 11-3 of reference 8. With R known, the fixed-end moments of each member can be found by the use of equation (7). These fixed-end moments can be balanced and distributed in the usual manner until the desired degree of accuracy of the secondary moments is obtained.

An example of this type of analysis without considering the effect of axial load, except for determining the change in length of each member, is given in Thompson and Cutler's discussion of Professor Cross' paper. (See reference 2.) The method is the same if axial load is considered except that graphs I to XI would be used in determining the various factors and fixed-end moments.

THE RECTANGULAR BENT

Figure 17 shows a single-story rectangular bent subjected to a side load S. If the two columns are considered free bodies, they are acted on by axial loads, shears, and end moments, as shown in figure 18. Part of the side load S is carried as shear in column AB and the rest in column CD. Hence $S_1 + S_2 = S$. Taking moments about A:

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} - S_1 h - S_2 h = 0$$

or

$$\Sigma M = h(S_1 + S_2) = Sh$$

This is known as the "bent equation" and states that the sum of the moments at the top and bottom of the columns of a story is equal to the shear on the story times the story height. This is an equation of equilibrium that must be satisfied in the analysis of all rectangular bents. The equation is valid for a bent of any number of columns and for any story of a multistory bent. In all cases the load S is the total shear on the story under consideration.

In order to analyze a rectangular bent by moment distribution it is first assumed that the horizontal beams are infinitely stiff. In this condition the structure is allowed to deflect laterally until the sum of the resisting moments at the ends of the columns becomes equal to the product of the shear and the story height.

In order to determine the effect of the deflection it is necessary to know how the resisting moments are divided among the columns. As the columns are connected by rigid horizontal beams, the deflections of all the columns are equal. (The change in length of the horizontal beams due to axial load is negligible when compared with the deflection of the columns due to bending.) With the horizontal beams assumed infinitely stiff, equation (7) may be applied to find the moments at the ends of the columns caused by the deflection. This gives:

$$M_{top} = M_{bottom} = \frac{EI\delta}{L^2 (2\beta - \alpha)} \quad \text{for each column.}$$

As E, L, and δ are the same for all columns, the resisting moment is seen to be divided in proportion to $I/(2\beta - \alpha)$. Therefore, the first step in the analysis of a rectangular bent is to divide the equilibrant of the story moment along the columns of each story in proportion to the value of $I/(2\beta - \alpha)$ for each column. These balancing moments are divided equally between the top and bottom of each column.

This division of moments satisfies the bent equation but leaves unbalanced moments at the joints of the structure. If there are any loads between panel points, the fixed-end moments caused by these also contribute to the unbalanced moments at the joints. In order to equilibrate the unbalanced moments it is necessary to assume that the horizontal beams lose their infinite rigidity and allow the joints to rotate until sufficient resisting moments are created. The balancing moments are distributed in proportion to the distribution factors of the members, and moments are carried over to the far ends of the members. During this step it is necessary to assume that the joints are restrained from translation in order that the expressions for distribution and carry-over factors may be applied. After these balancing and carry-over moments are applied, the bent equation is no longer satisfied. Therefore, the horizontal beams are again assumed infinitely stiff and the structure again allowed to deflect until the bent equation is again satisfied. The operations are continued until the error in the bent equation and the unbalanced moments at the joints are small enough to neglect.

EXAMPLE OF SINGLE-STORY RECTANGULAR BENT

Figure 19 shows a single-story rectangular bent subjected to a side load. The dimensions are given in the figure. $I_{AB} = I_{CD} = 2I_{BC}$. It is assumed that $L/j = 0$ for BC and CD and $L/j = 2.5$ for column AB. Although the side load will put axial compression in AB and BC and tension in DC, the exact amounts of these loads are unknown until the moments at the ends of the columns are determined. In the usual case the amounts of these axial loads are negligible, the only axial loads of large magnitude being due to vertical loads on the bent. However, for accurate analysis the structure may be analyzed a second time, using the moments found in the first analysis to correct the axial loads in the columns. The problem given here is to be considered merely as an example to illustrate the use of the system and not as illustrative of conditions that might be met in actual design.

As I is the same for both columns, the story moment equilibrant is divided between the columns in inverse proportion to $2\beta - \alpha$. The value of $2\beta - \alpha$ may be read from graph XI. It is found that

$$M_{AB} : M_{CD} = 1 : 1.123$$

That is, $1/(1 + 1.23) = 47.2$ percent of the story moment is resisted by moments at the ends of AB and 52.8 percent by moments at the ends of CD. As the moments at the top and bottom of each column are equal, the equilibrant of the story moment is divided 23.6 percent to the top and 23.6 percent to the bottom of AB and 26.4 percent to the top and 26.4 percent to the bottom of CD.

The shear load is 180 pounds and the story height 20 feet; therefore the story moment is $-180 \times 20 = -3,600$ lb.-ft. The equilibrant of this, 3,600 lb.-ft. is divided $3,600 \times 23.6$ percent = 850 lb.-ft. to the top and 850 lb.-ft. to the bottom of AB. Similarly $3,600 \times 26.4$ percent = 950 lb.-ft. is distributed to the top and 950 lb.-ft. to the bottom of CD.

This leaves an unbalanced moment of 850 lb.-ft. at joint B and 950 lb.-ft. at joint C. Before these can be balanced, the distribution factors of the members must be determined. I/L is constant for all the members, so the distribution factors are proportional to the coefficients found in graph II. These coefficients are 1.00 for BC and CD and 0.772 for AB. Therefore the distribution factors for joint B are $0.772/(1 + 0.772) =$ 43.5 percent for AB and 56.5 percent for BC. At joint C the distribution factors are 50 percent for each of the members. The carry-over factors of 0.5 for BC and CD and 0.731 for AB are found in graph I.

The balancing moments at B are therefore -850×43.5 percent = -370 lb.-ft. to BA and $-850 \times 56.5 = -480$ lb.-ft. to BC. At C the balancing moments ate $-950 \times$ 50 percent = -475 lb.-ft. to both CB and CD. $-370 \times 0.731 = -271$ lb.-ft. are carried over to A, -475×0.5 $= -238$ lb.-ft. are carried over to D and B, and $-480 \times 0.5 = -240$ lb.-ft. are carried over to C.

If the moments on the ends of the columns are added, it will be found that the sum is no longer equal to 3,600 lb.-ft. The values $-370-271-475-238=-1{,}354$ lb.-ft. have been added to the columns since the bent equation was satisfied. This quantity, $-1{,}354$ lb.-ft., is called ΔM and is found by totaling all the balancing moments and carry-over moments that have been added to the columns since the last time the bent equation was satisfied.

The quantity ΔM is treated exactly the same as the original story moment. The equilibrant $-\Delta M$ is divided among the columns in proportion to $I/(2\beta-\alpha)$, and the process continued until the unbalanced moments at the joints and the unbalanced story moment are small enough to be neglected.

In figure 19 four cycles have been completed. The resulting moments are $M_A = 910$ lb.-ft., $M_B = 800$ lb.-ft., $M_C = 809$ lb.-ft., and $M_D = 1{,}079$ lb.-ft. The moments acting on the columns are all positive, and their total is 3,598 lb.-ft. The error of 2 lb.-ft. in the story moment is negligible.

This same bent was analyzed in reference 4, assuming no axial load in the members. The results in this case were $M_A = M_D = 969$ lb.-ft. and $M_B = M_C = 831$ lb.-ft.

Examples of multistoried bents and bents subjected to unsymmetrical vertical loads have been given in reference 4. When axial load is included, the only difference is the use of graphs I to XI in determining the various factors and fixed-end moments. The principles involved in these two cases are the same as those in the single-story bent, and if these are thoroughly understood, there should be no difficulty in applying them to the more complex structures.

APPLICATION TO AIRPLANE FUSELAGE TRUSS

Figure 20 shows the central portion of the side truss of an airplane fuselage. The structure has been analyzed for the various conditions of loading required by the Department of Commerce, and the members have been selected, assuming a restraint coefficient of 2, with the assumption that each member is subjected to pure axial load.

As is usually the case in airplane fuselage trusses, however, some of the members have side loads applied between the panel points; and in the landing conditions there are concentrated moments applied at the points where the chassis members join the fuselage. The effect of these conditions will be determined by moment distribution and the margins of safety computed. If desired, the secondary moments due to joint translation may be included in the fixed-end moments. These will be small, however, and the refinement hardly justifies the amount of labor involved in computing them. They are not included in the example.

As most of the members in the central part of the fuselage are designed for three-point landing, this condition of loading will be used in the example. The load factor for this condition is 5.85.

Table A gives the physical properties of each member and the axial load in the three-point landing condition. The values of L/j have been computed and recorded in the table, the letter following the figure indicating tension or compression, and the carry-over and stiffness factors have been determined from graphs I and II. The distribution factors have been computed and recorded on the figure at each joint, and the carry-over factors have been written on each member.

In this airplane, four of the items of loading are attached to the longerons between panel points. Although the weights are applied at an angle of 14° to the thrust axis in the three-point landing condition, only the components of load perpendicular to the members are used in figuring the fixed-end moments. The components parallel to the members have a slight effect upon the axial loads, but this is small enough to be neglected. The loads that contribute to the fixed-end moments are:

1. Instruments -- a concentrated load of 20 pounds applied on 2U-4U, 13 inches from 2U.

2. Baggage -- a concentrated load of 100 pounds applied on 5L-6L, 14 inches from 5L.

3. Passengers -- two concentrated loads of 364 pounds each, one applied on 3L-4L, 27 inches from 3L and the other applied on 4L-5L, 21 inches from 4L.

4. Floor load -- a uniformly distributed load of 0.769 pound per inch extending from 3L to 5L.

The loads given are the basic loads. In order to find the design loads they must be multiplied by half the load factor of 5.85. (Half the factor is used as there are two side trusses, each carrying half the load.) The fixed-end moments have been computed, using the appropriate formulas and curves, and the results recorded on figure 20 at the ends of members 2U-4U, 3L-4L, 4L-5L, and 5L-6L.

In addition to these fixed-end moments there are two concentrated moments applied to the fuselage by the chassis. A counterclockwise moment of 7,520 in.-lb. is applied at 3L, and a counterclockwise moment of 11,160 in.-lb. is applied at 4L. These moments may be considered as fixed-end moments on the chassis members. A counterclockwise moment applied to a joint means a clockwise moment acting on the end of the member; so both of the above moments are positive. They are treated exactly the same as the other fixed-end moments at the joints. When these moments were computed in the chassis analysis of the airplane, it was assumed that the fuselage was a rigid, unyielding structure. This is not a true assumption as the fuselage joints are capable of rotation to a slight degree; hence the actual moments applied at 3L and 4L would probably be somewhat less than those given. A precise solution would involve a very complicated analysis as the chassis presents a three-dimensional problem, with the members capable of carrying torsion as well as bending; so no modification will be attempted here. The values given are probably very close to the actual values, and the error is believed to be small.

Joints 2U, 2L, 7U, and 7L have been assumed rigidly supported. This is obviously an erroneous assumption, but the error involved is small. Joints 2U and 2L are where the engine mount is attached. Since the structure forward of these joints is relatively rigid, the assumption of complete rigidity is probably nearly correct. Although in practice it might be necessary to analyze the entire structure aft of 2U-2L, for the purpose of this example it seems desirable to consider only that portion forward of 7U-7L. To do this, some assumption must be made at joints 7U and 7L. The joints might have been assumed pinned with as much justification as assuming rigid joints, or they might have been assumed as 50 percent rigid, as advocated by Bruhn in reference 6. The latter method appears to be the most accurate, but it would necessitate a new set of curves for stiffness factor, and so is hardly practical. Owing to the assumption adopted, the moments found in members 6U-7U, 6U-7L, and 6L-7L should not be expected to be as accurate as those farther forward, as the effect of an error at any joint is more noticeable on the members coming into that joint than on members that are farther removed.

With the fixed-end moments, distribution factors, and carry-over factors determined, the process of balancing the moments may be commenced. In the figure, the joints have been balanced in the order of the magnitude of their unbalanced moments, and the carry-over moments recorded as soon as a joint is balanced, as this method gives the most rapid convergence of results. The order of balancing was 4L, 3L, 5U, 6L, 5L, 4U, 6U, 4L, 5U, 5L, 6U, 4U, 4L, 5U, 3L, 6L, and 5L. As soon as a joint was balanced for the last time, no more moments were carried over to it, in order that the check of $\Sigma M = 0$ for each joint might be obtained. The totals of the moments at the joints are recorded in the figure. The moments obtained by applying the Hardy Cross method to this truss without correcting for the effect of axial loads are shown in references 4 and 5.

APPENDIX I

Definitions

1. Fixed-End Moments: The moments that exist at the ends of a loaded member when those ends are rigidly fixed against rotation are called the fixed-end moments of that member.

2. Stiffness Factors: A number proportional to the couple that must be applied at one end of a member to cause unit rotation of that end, both ends of the member being assumed to have no movement of translation, is called the stiffness factor of that member. The stiffness factor will depend on the degree of restraint of the opposite end of the member from that at which the couple is applied. In this paper two such cases are considered, that in which the far end is fixed against rotation and that in which the far end is free to rotate.

3. Distribution Factor: If a moment is applied at a joint where two or more members are rigidly connected, the distribution factor of each member is the percentage of the applied moment that is absorbed by that member. The distribution factors of the members at a joint are proportional to the stiffness factors of those members. The sum of the distribution factors of the members at any joint must equal unity.

4. Carry-Over Factor: If a beam is simply supported at one end and fixed at the other, and a moment is applied at the simply supported end, a moment is developed at the fixed end. The carry-over factor is the ratio of the moment at the fixed end to that at the simply supported end. For a member without axial load and with constant moment of inertia, the carry-over factor is 0.5.

5. Sign Convention:

1) A clockwise moment acting on the end of a member is positive. Consequently, a clockwise moment acting on a joint is negative. This is in agreement with the convention used in reference 1 at the left end of a member but opposite to that convention for the right end of a member. Great care must be taken to interpret correctly the signs of bending moments obtained in the moment-distribution analysis before proceeding to the determination

of bending moment between the ends of a member or computing margins of safety.

2) Upward forces and deflections are positive, and hence in agreement with the corresponding convention in reference 1.

3) Clockwise rotations of the straight line joining the ends of a member are positive, the reverse of the convention used in reference 1 for slope.

APPENDIX II

Moment-Area Proof of Carry-Over Factor Formula

The principle of moment areas states that the deflection of any point "a" on a beam from the tangent at any other point "b" is equal to the moment about "a" of the area under the M/EI diagram between "a" and "b". In figure 1 the tangent at B is horizontal; so the deflection of A with respect to this tangent is zero. Hence the moment about A of the area under the M/EI diagram of the beam is zero. The expression for moment, using the moment-distribution convention of signs, is:

$$M = \frac{-M_B - M_A \cos \frac{L}{j}}{\sin \frac{L}{j}} \sin \frac{x}{j} + M_A \cos \frac{x}{j}$$

where x is the distance from A. The moment of the area under the M/EI curve is therefore

$$\delta = \frac{1}{EI} \int_0^L Mx \, dx = 0$$

The value of M is substituted and the expression integrated, making use of the formulas

$$C \int x \sin \frac{x}{j} \, dx = Cj^2 \left(\sin \frac{x}{j} - \frac{x}{j} \cos \frac{x}{j}\right)$$

$$C \int x \cos \frac{x}{j} \, dx = Cj^2 \left(\cos \frac{x}{j} + \frac{x}{j} \sin \frac{x}{j}\right)$$

The above integral reduces to

$$\frac{M_B}{M_A} = + \frac{\alpha}{2\beta}$$

which is the same as that derived by the three-moment equation.

APPENDIX III

Moment-Area Proof of Stiffness-Factor Formula

The rotation in radians of any point on a beam from the tangent at any other point is equal to the area under the M/EI diagram between the two points. In figure 1 the tangent at B is horizontal; so the area under the M/EI curve of the beam gives the absolute rotation of A. The expression for moment is:

$$M = \frac{-M_B - M_A \cos \frac{L}{j}}{\sin \frac{L}{j}} \sin \frac{x}{j} + M_A \cos \frac{x}{j}$$

where x is the distance from A. The area under the M/EI curve is:

$$\theta = \frac{1}{EI} \int_0^L M \, dx$$

Substituting the value of M and integrating, making use of the formulas

$$C \int \sin \frac{x}{j} \, dx = -Cj \cos \frac{x}{j}$$

$$C \int \cos \frac{x}{j} \, dx = +Cj \sin \frac{x}{j}$$

gives the expression:

$$\theta = \frac{LM_A}{3EI} \left(\beta - \frac{\alpha^2}{4\beta} \right)$$

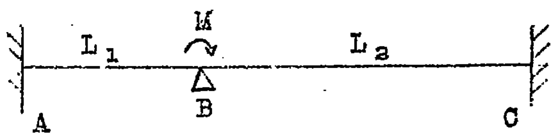
This may be written:

$$M_A = \frac{3EI\theta_A}{L\left(\beta - \frac{\alpha^2}{4\beta}\right)}$$

which is the same value as was found by using the Newell formula for slope.

APPENDIX IV

Check of Stiffness-Factor Formula by Three-Moment Equation



A continuous beam ABC has fixed ends at A and C. A clockwise moment M is applied at the center support B. As the carry-over factor is known to be $-\frac{\alpha}{2\beta}$, using the three-moment equation convention of signs, it is known that $M_A = -\left(\frac{\alpha_1}{2\beta_1}\right) M_{-B}$ and $M_C = -\left(\frac{\alpha_2}{2\beta_2}\right) M_{+B}$. The three-moment equation for spans 1 and 2 is:

$$\frac{M_A L_1 \alpha_1}{I_1} + \frac{2M_{-B}\ L_1 \beta_1}{I_1} + \frac{2M_{+B}\ L_2 \beta_2}{I_2} + \frac{M_C L_2 \alpha_2}{I_2} = 0$$

Substituting the known values of M_A and M_C in terms of M_{-B} and M_{+B} and reducing gives:

$$\frac{M_{-B}}{M_{+B}} = -\frac{\frac{L_2}{I_2}\left(\beta_2 - \frac{\alpha_2{}^2}{4\beta_2}\right)}{\frac{L_1}{I_1}\left(\beta_1 - \frac{\alpha_1{}^2}{4\beta_1}\right)} = -\frac{\frac{L_2}{I_2}\ \frac{(4\beta_2{}^2 - \alpha_2{}^2)}{3\beta_2}}{\frac{L_1}{I_1}\ \frac{(4\beta_1{}^2 - \alpha_1{}^2)}{3\beta_1}}$$

showing that in this special case the moment applied at B is distributed in proportion to the value of $\frac{I}{L}\frac{3\beta}{4\beta^2 - \alpha^2}$ of each member. (See equation (3).)

APPENDIX V

Carry-Over Factor for Axial Tension

Assume that the beam of figure 1 is subjected to axial tension. A is free to rotate, but restrained from transverse motion. A given moment M_A is applied at A and it is desired to find the magnitude of the resisting moment at B, M_B. The three-moment equation for this beam is:

$$\frac{M_1 L_1 \alpha_{h_1}}{I_1} + 2M_2 \left[\frac{L_1 \beta_{h_1}}{I_1} + \frac{L_2 \beta_{h_2}}{I_2}\right] + \frac{M_3 L_2 \alpha_{h_2}}{I_2} = 0$$

Where

$$M_1 = M_A$$

$$M_2 = M_B$$

$$L_2 = 0$$

this gives

$$\frac{M_A L \ \alpha_h}{I} + 2M_B \left(\frac{L\beta_h}{I}\right) = 0$$

whence

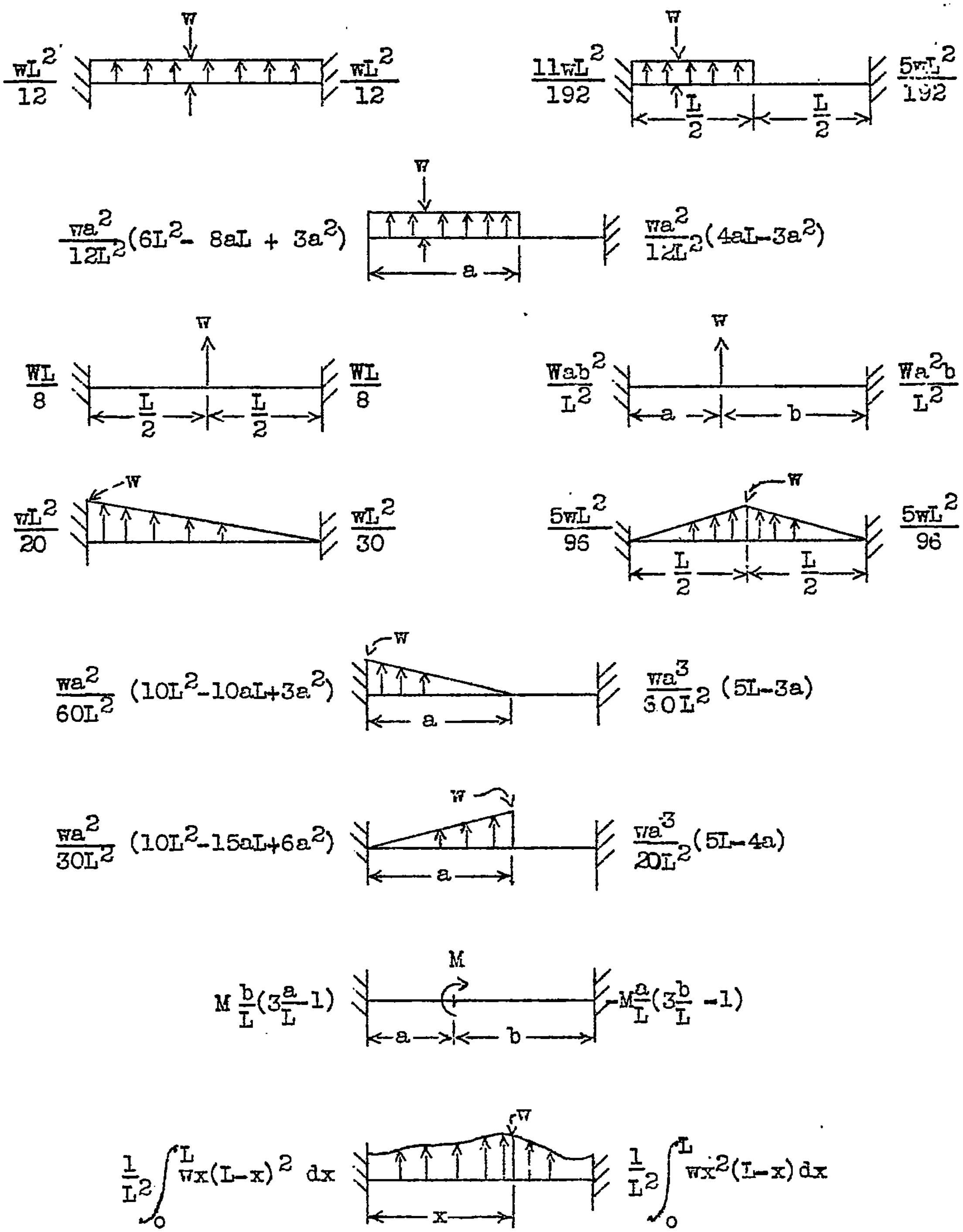
$$M_B = -\frac{\alpha_h}{2\beta_h} M_A$$

which is the same expression as that for compression except that α and β have been changed to α_h and β_h, respectively.

APPENDIX VI

Fixed-End Moments when $\frac{L}{j} = 0$. (See reference 3, p. 85.)

Note: In the moments given in the following sketches the convention of signs is the same as that used in the three-moment equations by Niles and Newell. For the convention used in moment distribution, the sign of the moments given at the right end must be changed.



$$\frac{1}{L^2}\int_0^L wx(L-x)^2\,dx \qquad \frac{1}{L^2}\int_0^L wx^2(L-x)\,dx$$

General case - Any loading

REFERENCES

1. Niles, Alfred S., and Newell, Joseph S.: Airplane Structures. John Wiley & Sons, Inc., 1929.

2. Cross, Hardy: Analysis of Continuous Frames by Distributing Fixed-End Moments. A.S.C.E. Trans., vol. 96, 1932, pp. 1-10. Discussion, pp. 11-156.

3. Cross and Morgan: Continuous Frames of Reenforced Concrete. John Wiley & Sons, Inc., 1932.

4. Williams, Harry A.: The Hardy Cross Method of Determining Moments in Statically Indeterminate Structures. Stanford University Thesis, 1933.

5. Williams, Harry A.: The Application of the Hardy Cross Method of Moment Distribution. Paper AER-56-9, A.S.M.E. Trans., May 1934.

6. Bruhn, E. F.: The Moment Distribution Method for Solving Continuous Frameworks. Aviation Engineering, March 1933, pp. 5-8.

7. Morris, Clyde T.: (Portion of Discussion, reference 2.) A.S.C.E. Trans., vol. 96, 1932, pp. 66-69.

8. Sutherland, Hale, and Bowman, Harry L.: An Introduction to Structural Theory and Design. John Wiley & Sons, Inc. 1st edition, 1930; 2d edition, 1935. (The second edition has a much more complete discussion of the Hardy Cross method than the first.)

TABLE A

Physical Properties of Fuselage Truss Members

Member	Size (a)	Length	Axial load	$\frac{L}{j}$	Carry-over factor	Stiff-ness factor coeffi-cient	Stiff-ness factor $\times 10^{4}$
		in.	lb.				in.^{3}
2U-4U	$1\frac{1}{4}$ X0.035	57.3	2815	3.59T	0.310	1.371	5.903
4U-5U	1 X .035	48.0	-2630	4.11C	3.27	.240	.619
5U-6U	1 X .035	36.1	-3950	3.79C	1.823	.391	1.340
6U-7U	$1\frac{1}{8}$ X .035	50.0	-4070	4.44C	23.740	.040	.143
2L-3L	1 X .035	39.3	-2110	3.01C	.925	.652	2.052
3L-4L	$1\frac{1}{4}$ X .035	39.0	-4325	3.03C	.936	.647	4.093
4L-5L	$1\frac{1}{4}$ X .035	26.0	8725	2.87T	.357	1.250	11.861
5L-6L	1 X .035	36.0	4355	3.97T	.289	1.442	4.955
6L-7L	1 X .035	51.0	3895	5.32T	.220	1.721	4.174
2U-3L	1 X .035	53.7	-1870	3.88C	2.50	.352	.811
3L-4U	$1\frac{1}{8}$ X .035	56.6	1265	2.80T	.362	1.239	3.901
4U-4L	$1\frac{1}{8}$ X .035	58.3	4710	5.56T	.208	1.773	5.419
4L-5U	$1\frac{1}{4}$ X .049	59.9	-2965	3.31C	1.120	.565	3.149
5U-5L	$1\frac{1}{4}$ X .049	54.0	4295	3.60T	.300	1.375	8.502
5U-6L	1 X .035	64.9	-845	3.15C	1.004	.614	1.171
6U-6L	$\frac{7}{8}$ X .035	57.0	220	1.74T	.435	1.098	1.572
6U-7L	$\frac{7}{8}$ X .035	68.7	105	1.45T	.453	1.069	1.270

(a) Diameter and thickness, inches.

TABLE B

$\frac{L}{j}$	Carry-over factor	Stiffness factor coefficient (fixed)	Stiffness factor coefficient (pinned)	Column distribution coefficient
		Axial compression		
0	0.50000	1.00000	0.75000	1.0000
1.0	.52640	.96628	.69852	1.0170
2.0	.62628	.85904	.52210	1.0737
2.5	.73097	.77193	.35947	1.1226
3.0	.91893	.65605	.10206	1.1915
3.5	1.31574	.50201	-.36705	1.2903
4.0	2.56030	.29388	-1.6294	1.4364
4.5	-	.00479	-	1.6691

Axial tension

0	0.50000	1.0000	0.7500	1.0000
1.0	.47625	1.0329	.7986	.9837
2.0	.41737	1.1268	.9305	.9392
3.0	.34768	1.2703	1.1167	.8762
4.0	.28419	1.4492	1.3321	.8060
5.0	.23308	1.6519	1.5622	.7364
6.0	.19405	1.8706	1.7999	.6716

TABLE C

Fixed-End Moment Coefficients

$\frac{L}{j}$	Uniform load entire span	Uniformly varying load		Concentrated load midspan
		C_A	C_B	
		Axial compression		
0	12.000	30.000	20.000	8.000
1.0	11.798	29.396	19.713	7.832
2.0	11.176	27.655	18.755	7.322
2.5	10.690	26.291	18.015	6.930
3.0	10.071	24.560	17.072	6.441
3.5	9.301	22.436	15.889	5.846
4.0	8.338	19.819	14.417	5.127
4.5	7.190	16.766	12.588	4.301
		Axial tension		
0	12.000	20.000	20.000	8.000
1.0	12.198	30.577	20.294	8.165
2.0	12.779	32.221	21.178	8.657
3.0	13.695	34.885	22.541	9.448
4.0	14.888	38.370	24.329	10.504
5.0	16.297	42.523	26.431	11.790
6.0	17.864	47.167	28.743	13.256

TABLE D

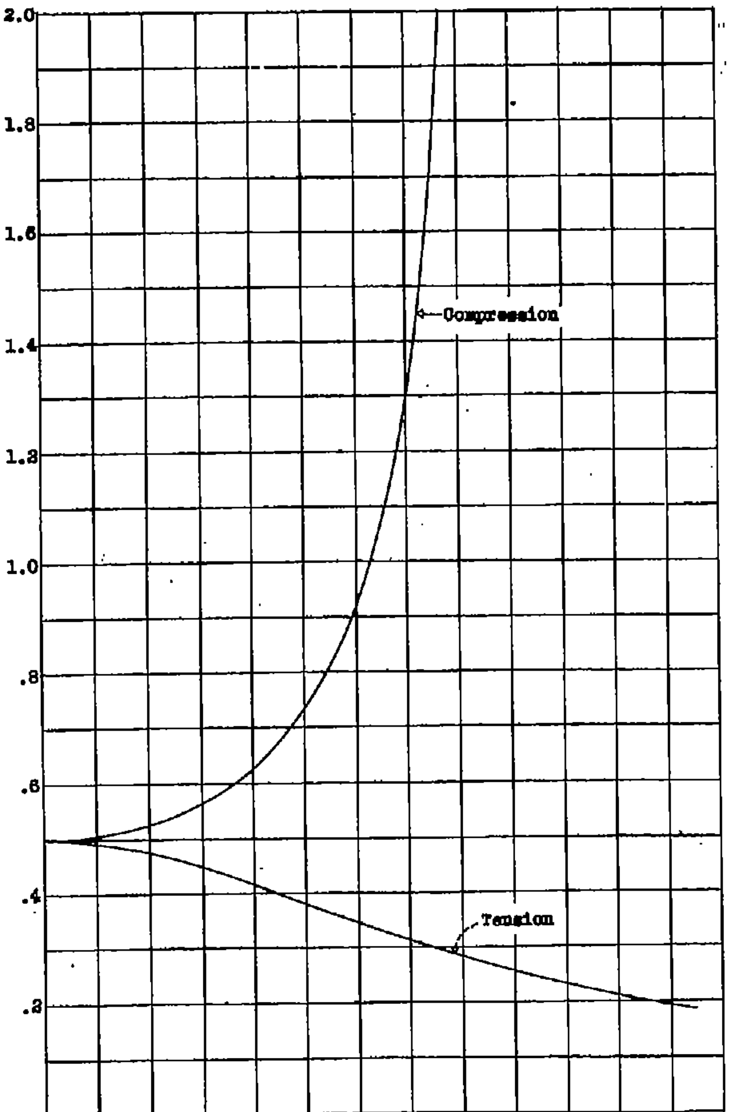
Fixed-End Moment Coefficients
Concentrated Load at Any Point on the Span

$\frac{L}{j}$	$\frac{a}{L} = 0.1$	$\frac{a}{L} = 0.2$	$\frac{a}{L} = 0.3$	$\frac{a}{L} = 0.4$	$\frac{a}{L} = 0.5$	$\frac{a}{L} = 0.6$	$\frac{a}{L} = 0.7$	$\frac{a}{L} = 0.8$	$\frac{a}{L} = 0.9$
				Axial compression					
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0	1.0066	1.0118	1.0160	1.0191	1.0216	1.0230	1.0222	1.2015	1.0172
2.0	1.0264	1.0491	1.0679	1.0824	1.0926	1.0983	1.0994	1.0961	1.0884
2.5	1.0431	1.0807	1.1122	1.1368	1.1544	1.1646	1.1670	1.1617	1.1487
3.0	1.0657	1.1242	1.1734	1.2135	1.2420	1.2590	1.2646	1.2557	1.2366
3.5	1.0964	1.1841	1.2604	1.3226	1.3685	1.3965	1.4061	1.3960	1.3669
4.0	1.1412	1.2720	1.3888	1.4863	1.5604	1.6075	1.6255	1.6136	1.5724
4.5	1.2018	1.3987	1.5805	1.7372	1.8600	1.9420	1.9780	1.9399	1.9082
				Axial tension					
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0	.9953	.9885	.9839	.9812	.9798	.9775	.9805	.9825	.9654
2.0	.9767	.9579	.9429	.9316	.9241	.9206	.9198	.9231	.9320
3.0	.9514	.9127	.8828	.8611	.8467	.8399	.8400	.8469	.8612
4.0	.9206	.8595	.8140	.7819	.7616	.7520	.7537	.7656	.7887
5.0	.8870	.8039	.7441	.7033	.6785	.6683	.6717	.6891	.7214
6.0	.8530	.7493	.6778	.6308	.6035	.5933	.5992	.6219	.6627

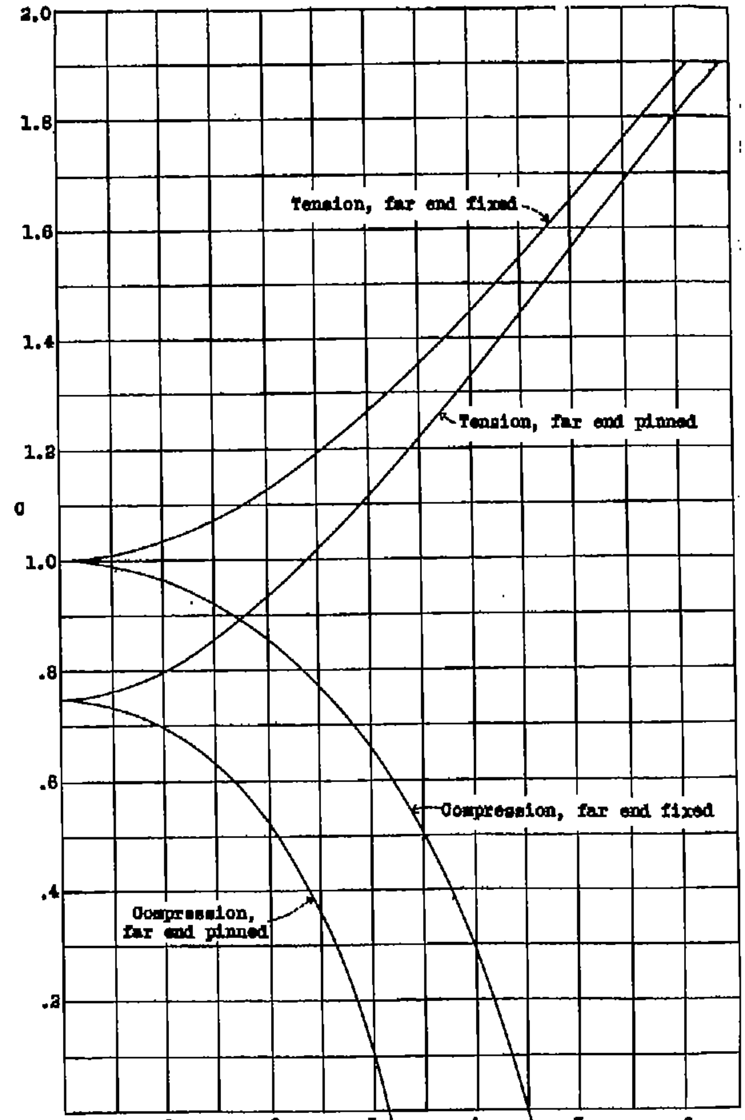
TABLE E

Fixed-End Moment Coefficients
Uniformly Distributed Load Over Part of Span

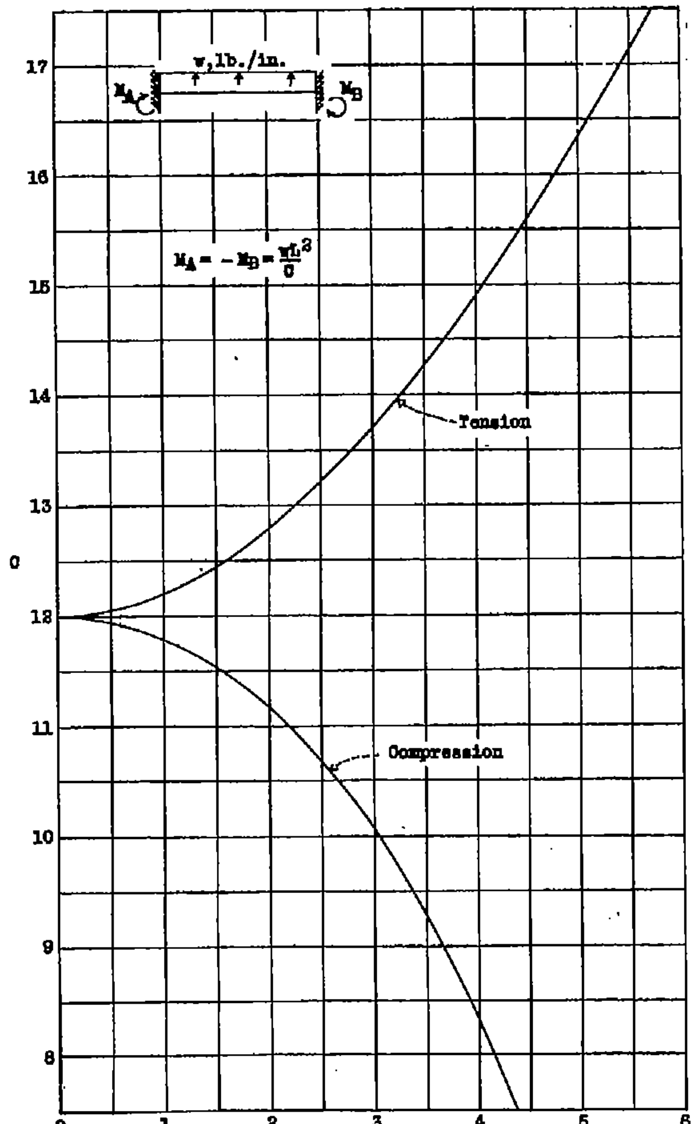
	Values of M_A					Values of M_B				
$\frac{L}{j}$	$\frac{a}{L}=0.1$	$\frac{a}{L}=0.2$	$\frac{a}{L}=0.3$	$\frac{a}{L}=0.4$	$\frac{a}{L}=0.5$	$\frac{a}{L}=0.1$	$\frac{a}{L}=0.2$	$\frac{a}{L}=0.3$	$\frac{a}{L}=0.4$	$\frac{a}{L}=0.5$
Axial compression										
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0	1.0053	1.0095	1.0114	1.0133	1.0148	1.0123	1.0085	1.0180	1.0215	1.0226
2.0	1.0174	1.0329	1.0453	1.0555	1.0370	1.0681	1.0922	1.0963	1.0976	1.0970
2.5	1.0282	1.0536	1.0746	1.0915	1.1044	1.1500	1.1554	1.1620	1.1643	1.1625
3.0	1.0438	1.0824	1.1152	1.1420	1.1626	1.2254	1.2444	1.2552	1.2585	1.2551
3.5	1.0637	1.1210	1.1706	1.2117	1.2436	1.3575	1.3815	1.3961	1.3994	1.3925
4.0	1.0935	1.1780	1.2526	1.3153	1.3648	1.5569	1.5939	1.6133	1.6161	1.6029
4.5	1.1320	1.2573	1.3708	1.4684	1.5470	1.8860	1.9383	1.9646	1.9635	1.9376
Axial tension										
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.0	.9939	.9921	.9895	.9877	.9864	.9860	.9851	.9793	.9750	.9751
2.0	.9840	.9717	.9611	.9528	.9470	.9408	.9267	.9225	.9214	.9217
3.0	.9678	.9407	.9192	.9033	.8915	.8656	.8539	.8470	.8427	.8428
4.0	.9461	.9037	.8708	.8459	.8282	.7931	.7772	.7644	.7579	.7571
5.0	.9234	.8641	.8204	.7867	.7640	.7321	.7049	.6746	.6772	.6752
6.0	.8993	.8248	.7697	.7304	.7033	.6756	.6421	.6190	.6060	.6023



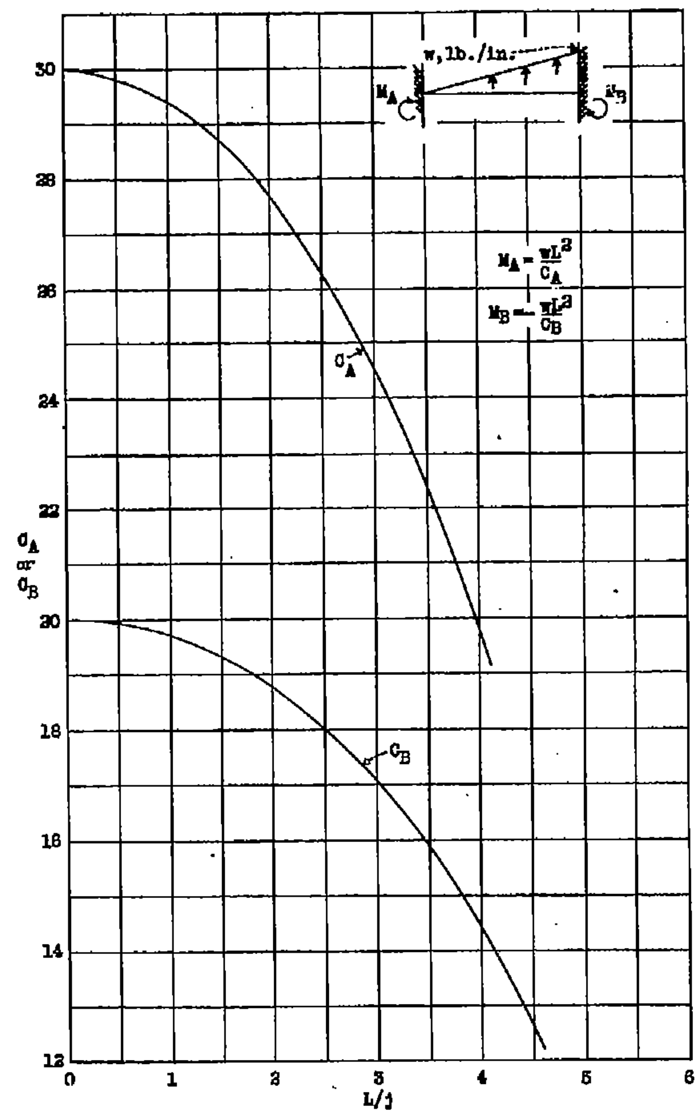
Graph I.- Carry-over factor. $C = \frac{\alpha}{2\beta}$



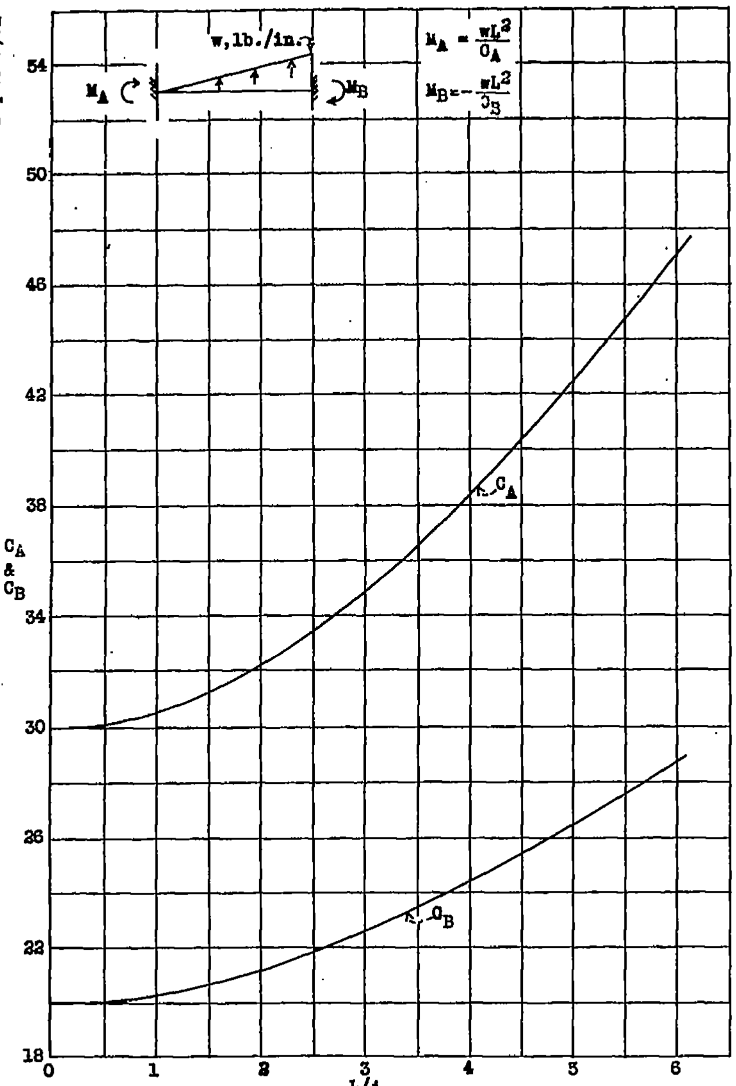
Graph II.- Stiffness factor coefficient. $K = C \frac{I}{L}$



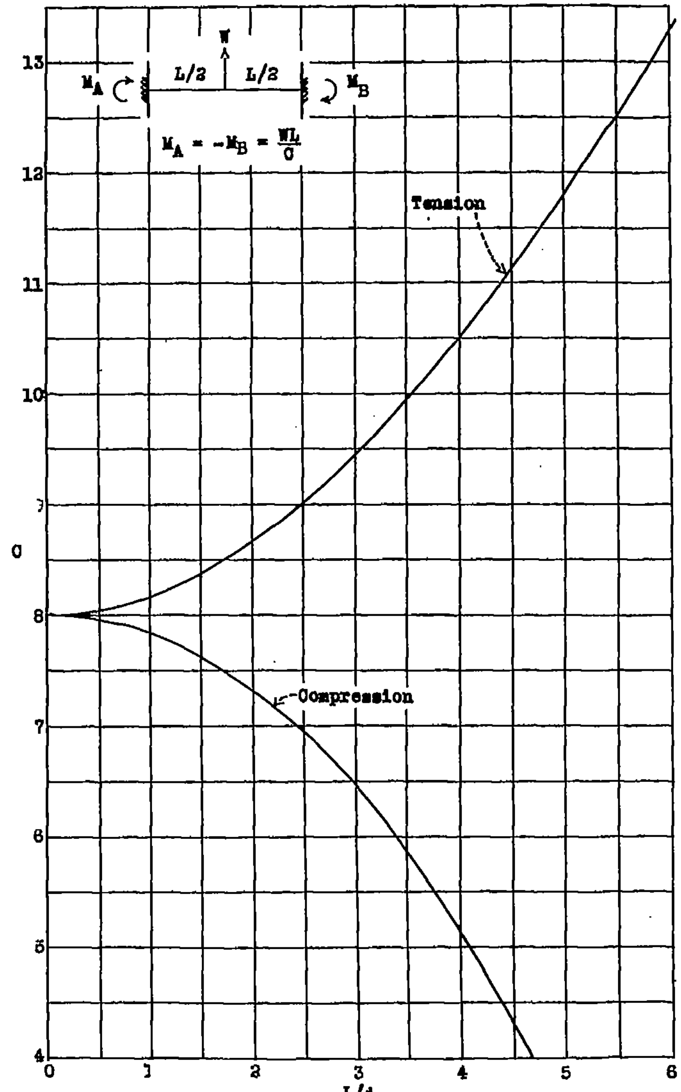
Graph III.- Fixed-end moment coefficient. Uniform load.



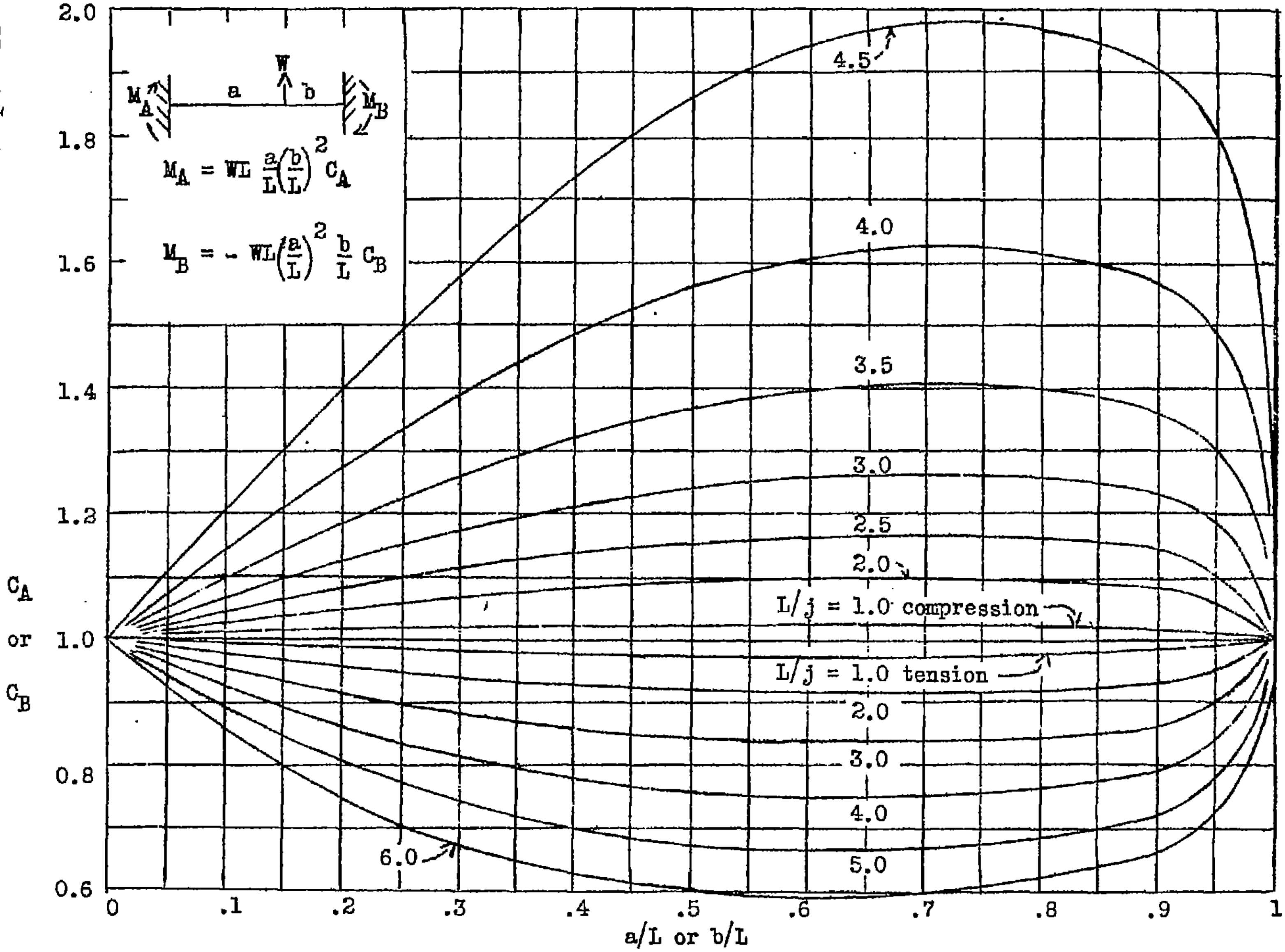
Graph IV.- Fixed-end moment coefficient. Uniformly varying load. Axial compression.



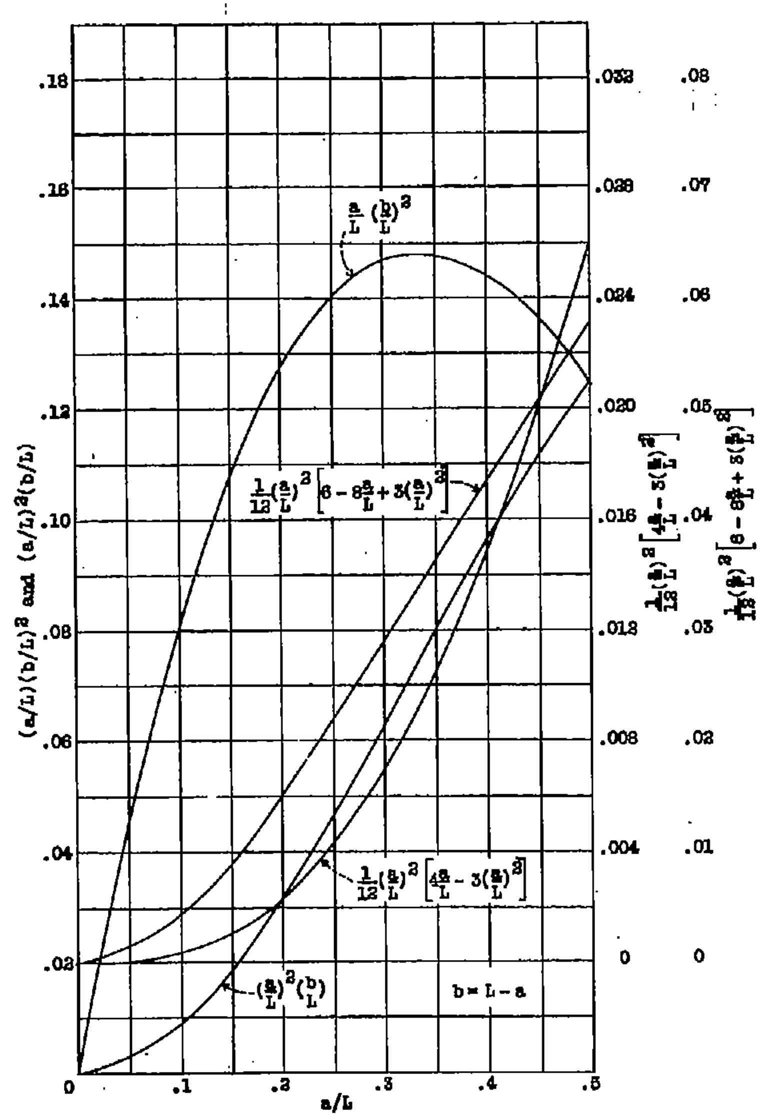
Graph V.-Fixed-end moment coefficient. Uniformly varying load. Axial tension.



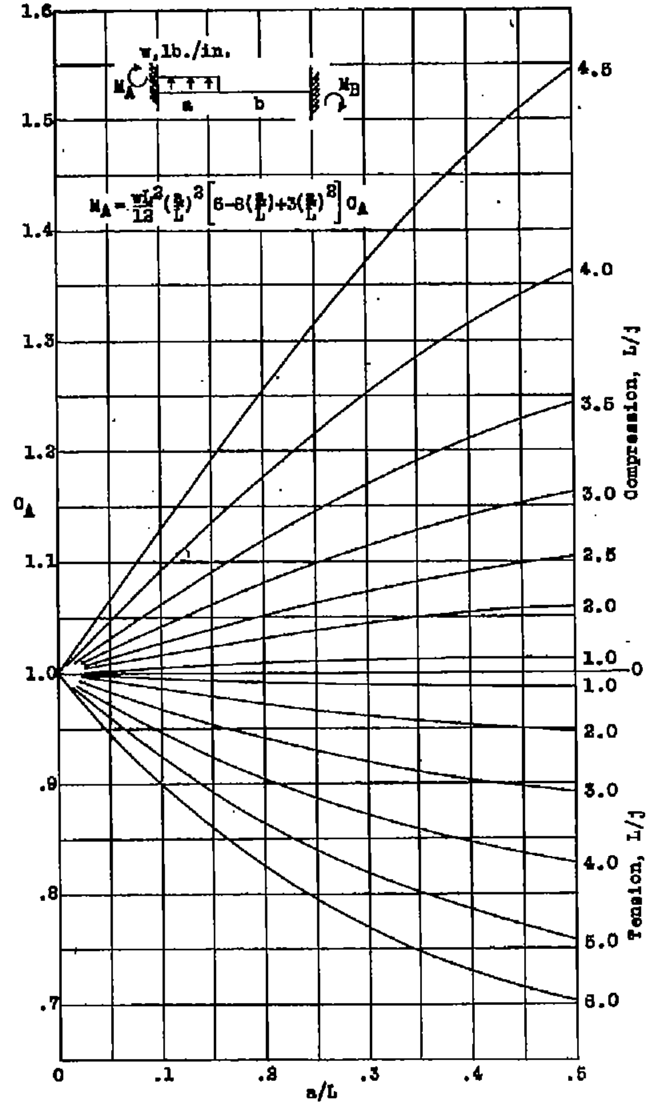
Graph VI.-Fixed-end moment coefficient. Concentrated load at mid-span.



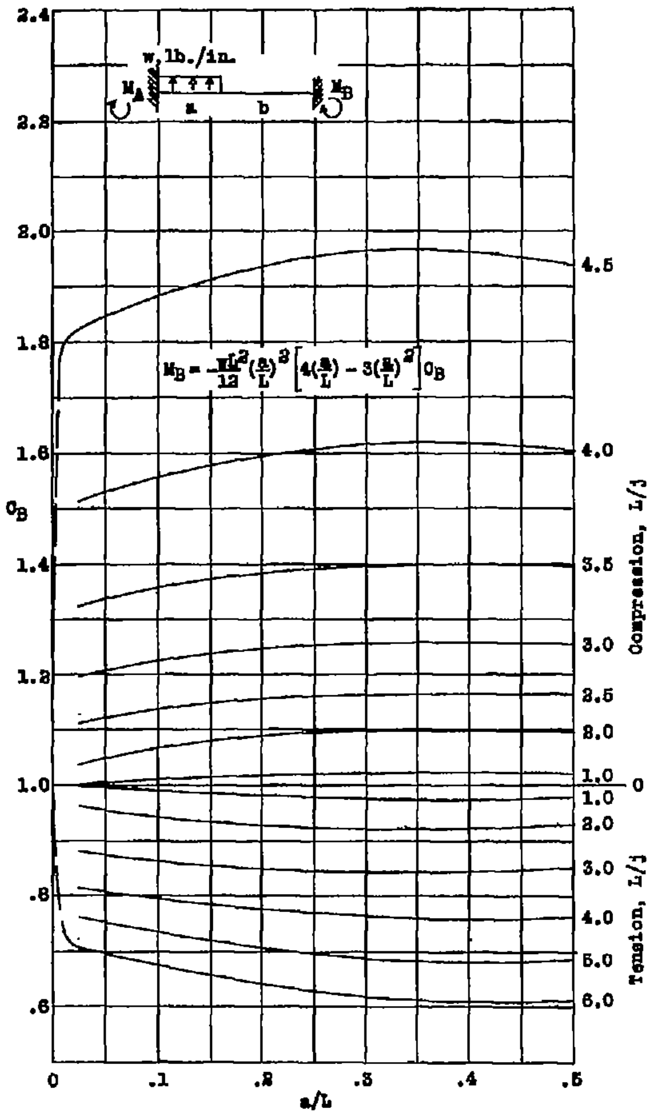
Graph VII. Fixed-end moment coefficient. Concentrated load



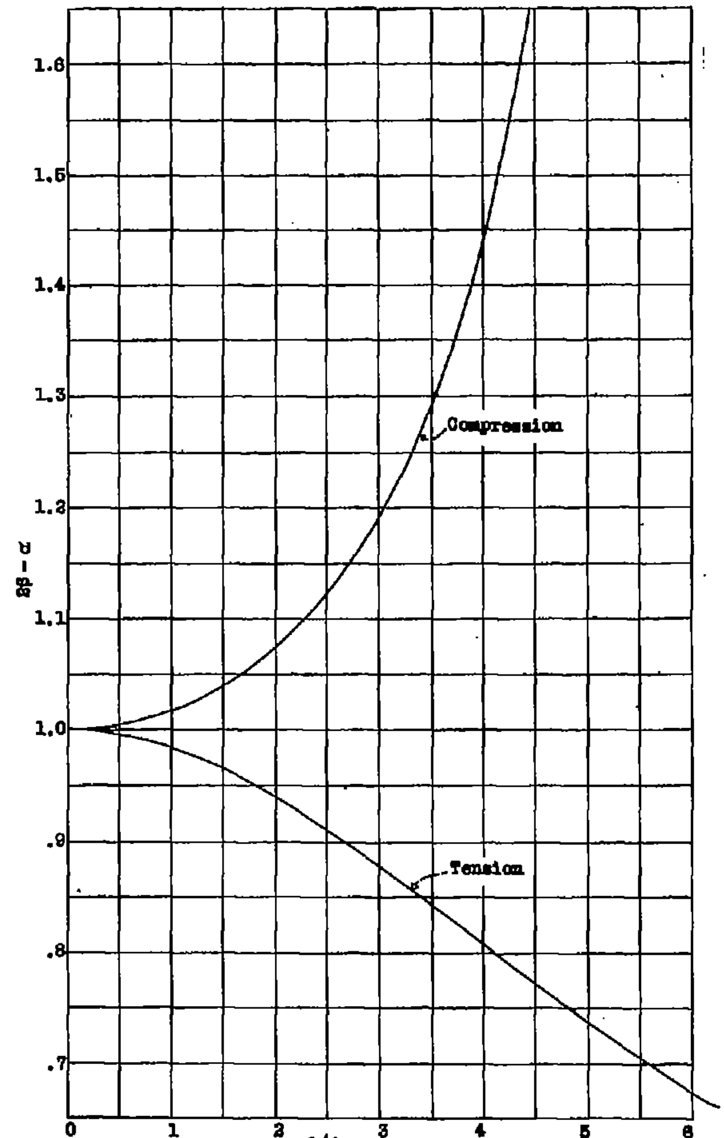
Graph VIII



Graph IX.- Fixed-end moment coefficient for M_A. Uniform load of part of the span.



Graph X.- Fixed-end moment coefficient for M_B.
Uniform load over part of the span.



Graph XI.- Column distribution factor, $2\beta - \alpha$.

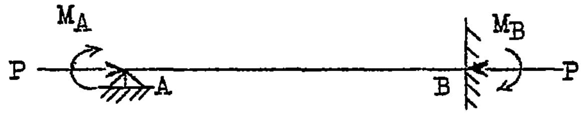


Figure 1.

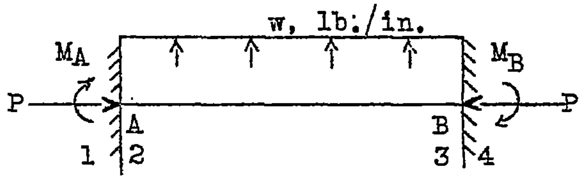


Figure 2.

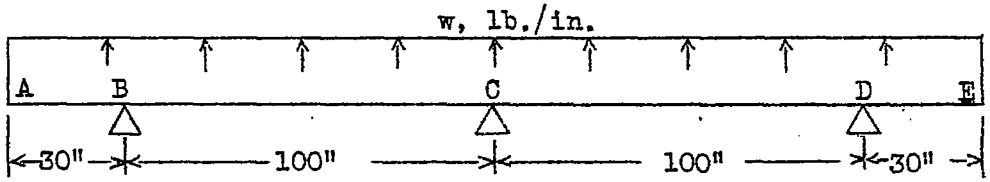
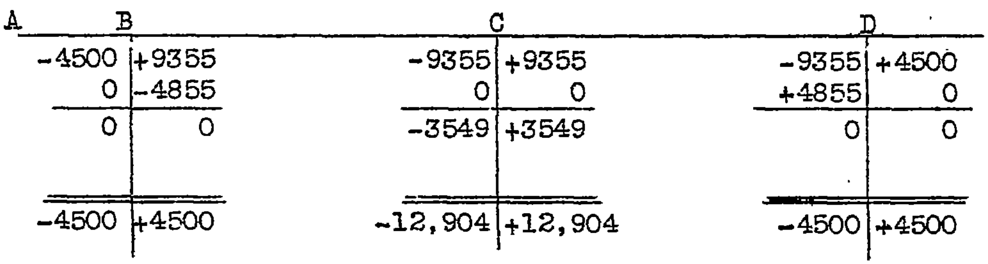


Figure 3.



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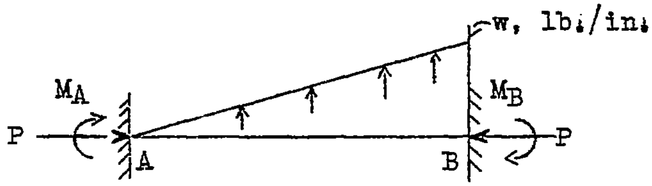

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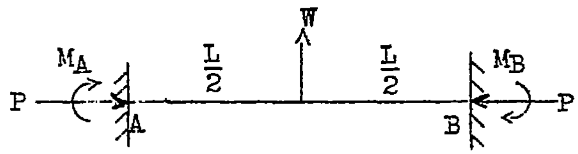

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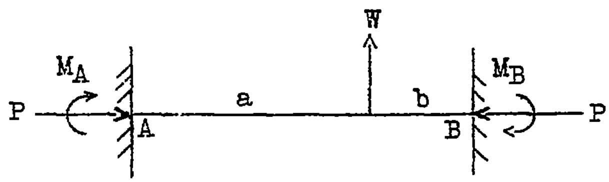

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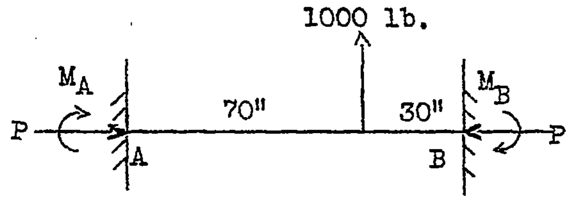

Figure 8.

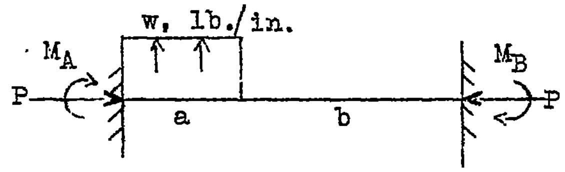

Figure 9.

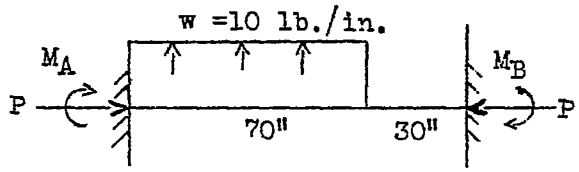

Figure 10.

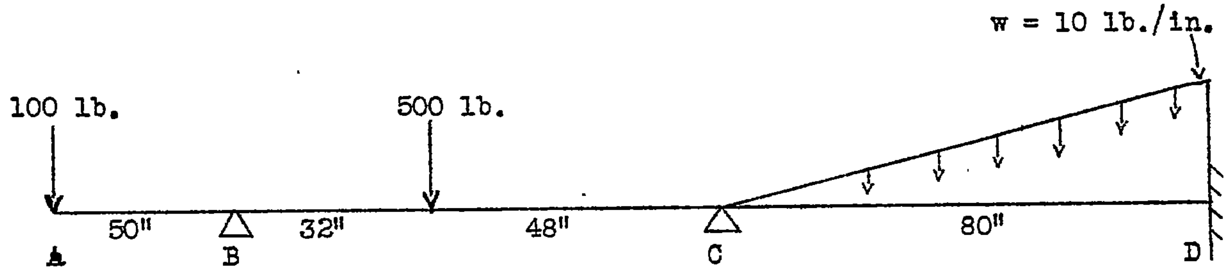

Figure 11.

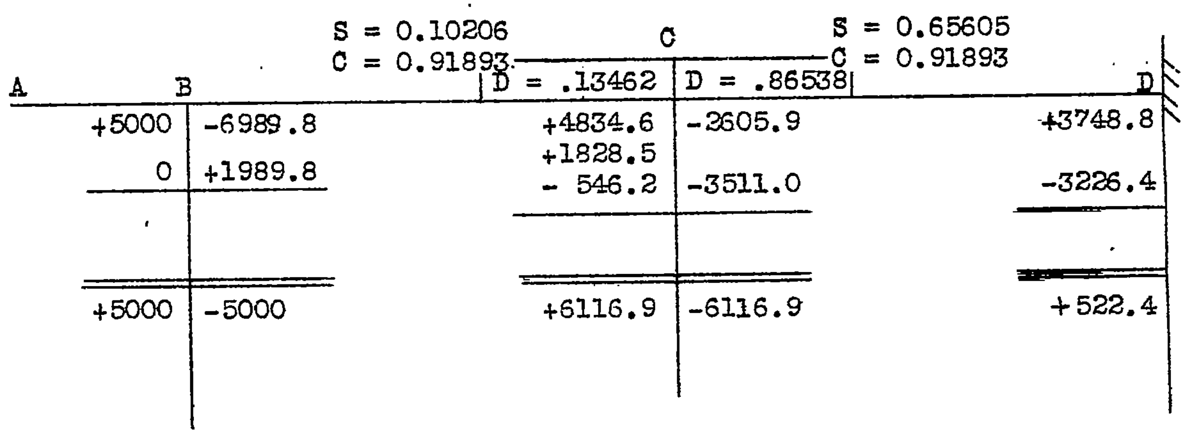

Figure 12.

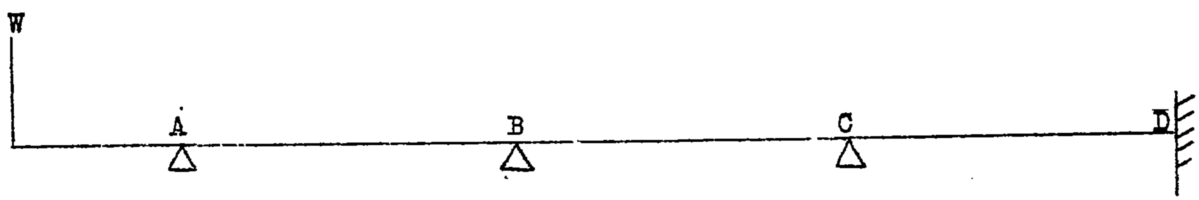

Figure 13.

	A D = 0	A D = 1.00	(C=0.9189) B D=.1346	B D=.8654	(C=0.9189) C D=.50	C D=.50	(C=0.9189) D
	+1000 0	-1000					
			-918.9 +123.7	0 +795.2			
					+730.6 -365.3	0 -365.3	
			0 +45.2	-335.7 +290.5			-335.7 0
					+266.9 -133.5	0 -133.5	
			0 +165.0	-122.7 +106.2			-122.7 0
					+ 97.6 - 48.8	0 -48.8	
			0 + 6.0	-44.8 +38.8			- 44.8 0
					+ 35.7 - 17.8	0 -17.8	
Total after 9 cycles			-727.5	+727.5	+565.4	-565.4	-503.2
Total after 18 cycles			-724.1	+724.1	+575.4	-575.4	-528.8
Three-mom. equation			-724.0	+724.0	+575.7	-575.7	-529.1
Neglecting axial load			-268.6	+268.6	+ 74.2	- 74.2	- 37.1

Figure 14.

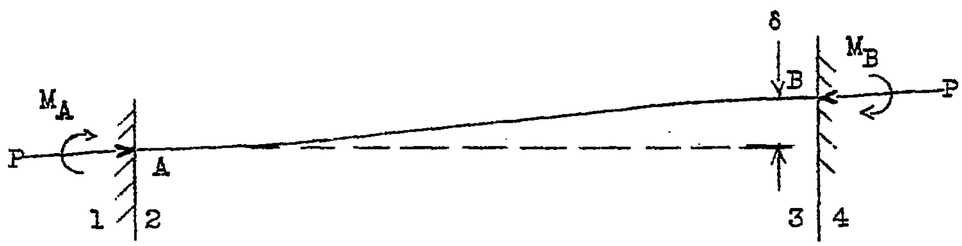

Figure 15.

A	B	(S=0.10206, C=0.91893) C D=.13462	C D=.86538	(S=0.65605, C=0.91893) D
+5000	-6989.8	+4834.6	-2605.4	+3748.8
	-3650.9	-3650.9	+3650.9	+3650.9
	+5640.7	+5183.4		
		- 997.8	-6414.3	-5894.3
+5000	-5000	+5364.3	-5364.3	+1505.4

Figure 16.

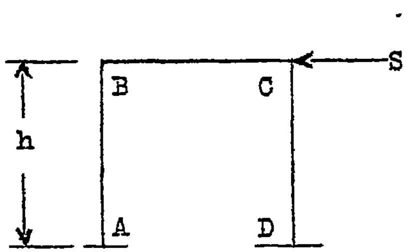

Figure 17.

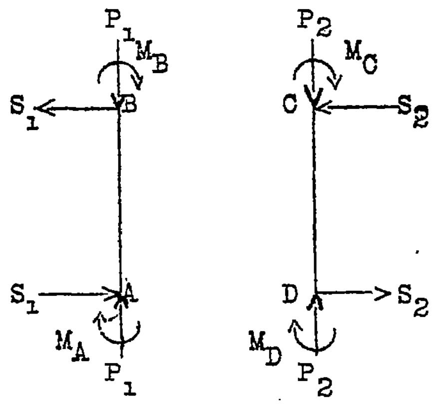

Figure 18.

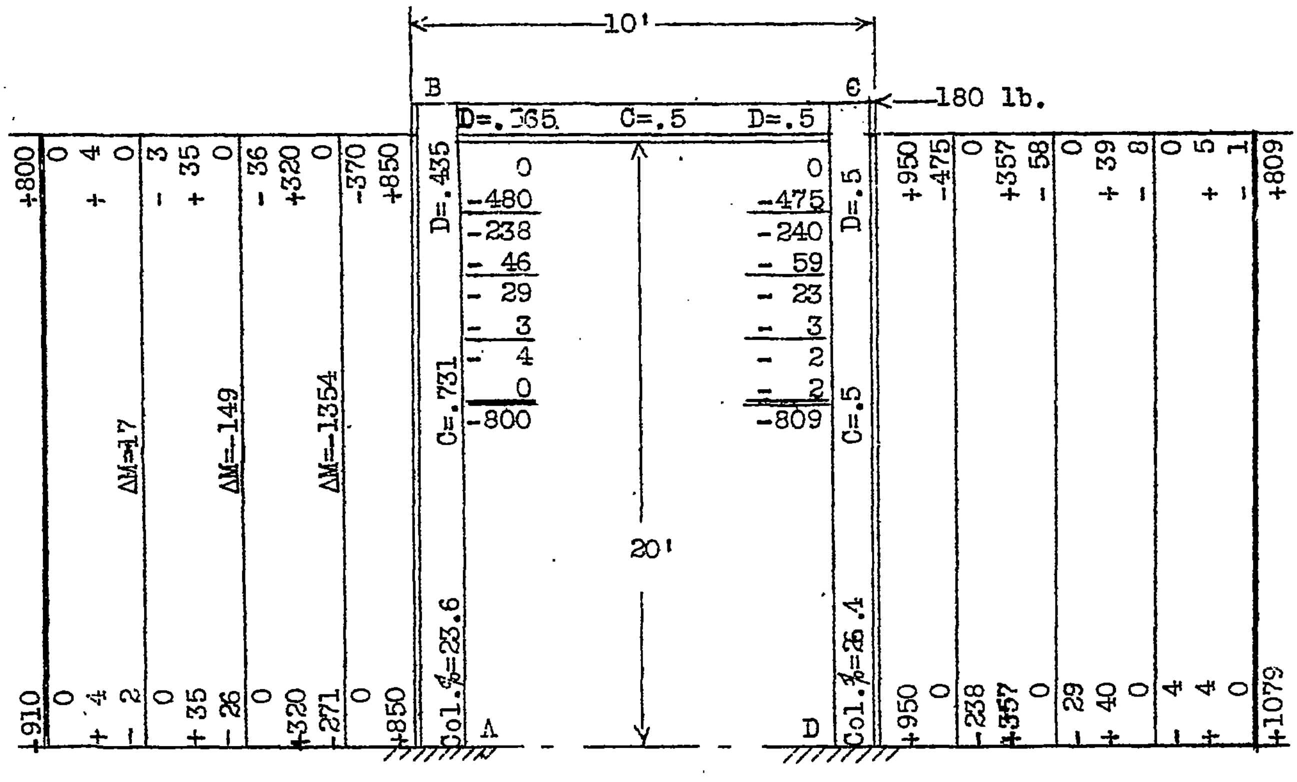

Figure 19.

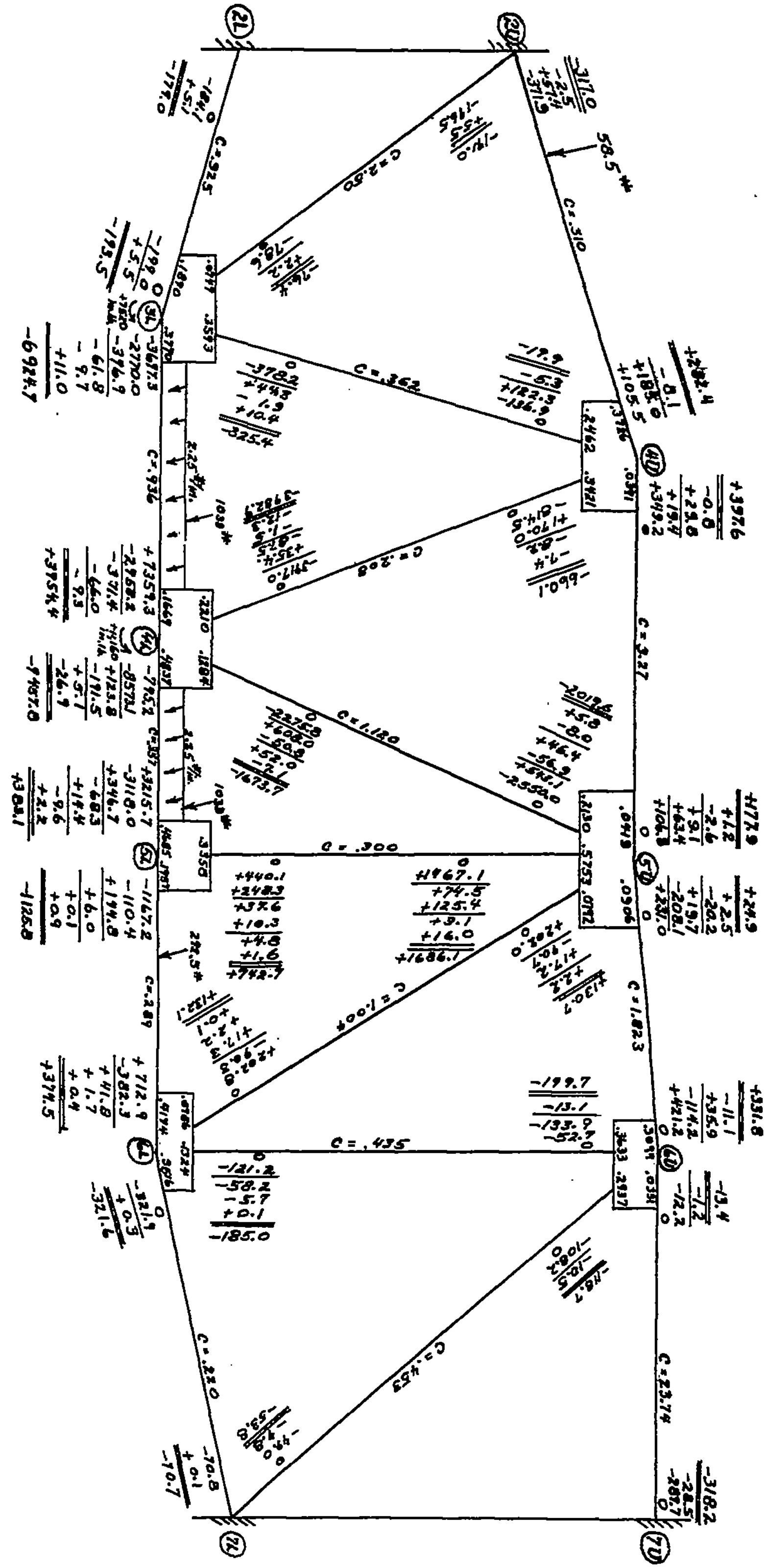

Fig. 20